Constraints in Universal Algebra

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Lecture 3
Outline

Lecture 1: Intersection problems and congruence $SD(\wedge)$ varieties

Lecture 2: Constraint problems in ternary groups (and generalizations)

Lecture 3: Constraint problems in Taylor varieties
WARNING

This lecture has been modified to fit our shorter attention spans.
Review

An **instance of 3-CSP(A)** of degree \( n \) is a list \((s_1, C_1), \ldots, (s_p, C_p)\) where
- Each scope \( s_i \) satisfies \( s_i \subseteq \{1, 2, \ldots, n\} \) and \( 1 \leq |s_i| \leq 3 \).
- Each constraint relation \( C_i \) is a non-empty subuniverse of \( A^{s_i} \).

It is **3-minimal** if
- Every 3-element subset of \( \{1, 2, \ldots, n\} \) occurs as a scope.
- For any two constraints \((s, C_i), (t, C_j)\), if \( s \subseteq t \) then \( \text{proj}_s(C_j) = C_i \).

The **solution-set** of the instance is \([s_1, C_1] \cap \cdots \cap [s_p, C_p]\), where

\[
[s_i, C_i] = \{a \in A^n : \text{proj}_{s_i}(a) \in C_i\} \leq A^n.
\]

**3-CSP(A):** Given a (3-minimal) instance, does a solution exist?

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**Central problem of CSP (Feder, Vardi) – Dichotomy**

Given a finite algebra \( A \), either

1. Find a poly-time algorithm deciding 3-minimal instances of 3-CSP(A),

or

2. Show that 3-CSP(A) is NP-complete.
3 algorithms deciding 3-minimal instances of 3-CSP(\(A\))

- Few subpowers algorithm
- Always answer “yes”

Gaussian Elimination

- Sets
- CM
- CD
- SD(\(\wedge\))
- 3-ary abelian groups of prime exponent
- Lattices
- Semilattices
Algebraic CSP Dichotomy Conjecture

There is a class, outside of which each 3-CSP($A$) is provably NP-complete.

**Conjecture** (Bulatov *et al*): For every $A$ inside the class, 3-CSP($A$) is in P.
Defining the “dividing line”

**Definition.** A term \( t(x_1, \ldots, x_n) \) is a **Taylor term** (for an algebra) if

1. It is idempotent (i.e., \( t(x, x, \ldots, x) = x \)).
2. \( n \geq 2 \).
3. For each \( 1 \leq i \leq n \) there is an identity satisfied by \( t \) of the form

   \[
   t(\ldots, x, \ldots) = t(\ldots, y, \ldots)
   \]

   where \( x \) occurs at position \( i \) on the left, \( y \) occurs at position \( i \) on the right, and all other positions are filled with \( x \) or \( y \).

**Example:** A Maltsev term is a Taylor term, because \( m(x, x, x) = x \) and

\[
\begin{align*}
m(x, x, y) &= m(y, y, y) \quad \text{works for } i = 1, 2 \\
m(x, x, x) &= m(x, y, y) \quad \text{works for } i = 3
\end{align*}
\]
Theorem/Conjecture (Bulatov, Jeavons, Krokhin 2005)

Let \( A \) be a finite, idempotent algebra.

1. (Theorem) If \( A \) does not have a Taylor term, then \( 3\text{-CSP}(A) \) is NP-complete.
2. (Conjecture) Otherwise \( 3\text{-CSP}(A) \) is in P.
Goals of this lecture:

1. Describe a new, “easy” poly-time CSP algorithm for ternary groups.
   - Roughly speaking, “enforcing 3-minimality + Gaussian elimination.”

2. Describe how the algorithm adapts to any Taylor algebra!

3. Caveats
   - The algorithm is for 2-CSP(\textbf{A}) only. (Which is fine.)
   - I don’t know whether the algorithm actually works . . .
First, some technicalities
Potatoes

Let \( Inst = ((s_1, C_1), \ldots, (s_p, C_p)) \) be a 3-minimal instance of 3-CSP(\( A \)), of degree \( n \).

1. \( V := \{1, 2, \ldots, n\} \). ("variables")

2. Every 3-element subset \( s \subseteq V \) is the scope of a \( unique \) constraint. Call it \( (s, C_s) \).

3. For all \( t \subseteq V \) with \( |t| = 1 \) or \( 2 \), there is a unique "implied" constraint \( (t, C_t) \), namely \( (t, \text{proj}_t(C_s)) \) for any \( t \subseteq s \) with \( |s| = 3 \).

4. Each \( C_{\{i\}} \) is a subuniverse of \( A \). The corresponding subalgebra is denoted \( P_i \) and is called a "potato."

\[ \begin{array}{l}
\text{C}_{\{1,2\}} \\
\text{P}_1 \\
\text{P}_2 \\
\text{P}_3 \\
\ldots \\
\end{array} \]

\[ \begin{array}{l}
\text{A} \\
\text{A} \\
\text{A} \\
\end{array} \]
Congruence completeness

**Definition.** A 3-minimal instance of 3-CSP($A$) is **congruence complete** if for every $i \in V$ and every $\alpha \in \text{Con}(P_i)$ there exists $j \in V$ such that $C\{i,j\}$ is the graph of a surjective homomorphism $h_{ij} : P_i \rightarrow P_j$ with kernel $\alpha$.

(I will say $P_j \sim P_i / \alpha$.)

We can always enforce congruence completeness (by adding new variables).
Now we focus on ternary groups
Definition. Let \( A = (G, xy^{-1}z) \) be a ternary group, \( \alpha \in \text{Con}(A) \), and \( p \) a prime. We say \( \alpha \) is an \textbf{elementary} \( p \)-\textbf{abelian} congruence if \( N := 1/\alpha \) (\( \triangleleft G \)) is an abelian group of exponent \( p \).

**Key fact.** If \( \alpha \in \text{Con}(A) \) is elementary \( p \)-abelian, then every \( \alpha \)-block \( C \), considered as a subalgebra \( C \leq A \), is a ternary abelian group of exponent \( p \).

**Proposition**

Let \( A = (G, xy^{-1}z) \) be a finite ternary group and \( \alpha \) a \textbf{minimal} congruence. If \( \alpha \) is abelian, then \( \alpha \) is elementary \( p \)-abelian for some prime \( p \).

**Proof.** \( N = 1/\alpha \) is a minimal normal subgroup of \( G \) and is abelian. If \( \exp(N) = mk \) is composite, then \( \{x \in N : mx = 1\} \) is a proper nontrivial subgroup of \( N \).

It is also characteristic in \( N \), so is normal in \( G \), contradiction. \( \square \)
Warning: technicalities ahead
Let $Inst$ be a 3-minimal instance of 3-CSP$(\mathbf{A})$, of degree $n$.

**Definition.** For each prime $p$, let

$$VC_p = \{ (i, \alpha) : i \in V, \alpha \in \text{Con}(P_i), \text{ and } \alpha \text{ is elementary } p\text{-abelian} \}.$$ 

For $(i, \alpha), (j, \beta) \in VC_p$, define $[(i, \alpha) \leq (j, \beta)]$ iff

$$((a, b), (a', b') \in C_{\{i,j\}} \& (a, a') \in \alpha) \implies (b, b') \in \beta.$$

($C_{\{i,j\}}$ “induces” a homomorphism $P_i/\alpha \to P_j/\beta$.)
Fix $p$ and $(i, \alpha) \in VC_p$. Define

$$V_{(i, \alpha)} = \{ j \in V : \exists \beta \in \text{Con}(P_j) \text{ with } (i, \alpha) \leq (j, \beta) \in VC_p \}.$$  

**Fact:** for each $j \in V_{(i, \alpha)}$ there exists a smallest witnessing $\beta$; call it $\beta_j$.

Let $f_j : P_i/\alpha \to P_j/\beta_j$ be the homomorphism induced by $C\{i,j\}$.

**Definition**

Let $A$, $\text{Inst}$, $p$ and $(i, \alpha)$ be as above.

1. $\text{Inst}_{(i, \alpha)}$ is the restriction of $\text{Inst}^{a}$ to the variable-set $V_{(i, \alpha)}$.
2. For each $\alpha$-block $B$, $\text{Inst}_{(i, \alpha)}^{B}$ is the restriction of $\text{Inst}_{(i, \alpha)}$ obtained by
   - Replacing each potato $P_j$ by $f_j(B)$, and
   - Restricting the constraint relations of $\text{Inst}_{(i, \alpha)}$ to these new potatoes.

\[a\text{More precisely, of the implied constraints } (t, C_t), 1 \leq |t| \leq 3, \text{ of } \text{Inst}.\]

**Note:** Each potato of $\text{Inst}_{(i, \alpha)}^{B}$ is a ternary abelian group of exponent $p$. 

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Lemma

Suppose \( s \subseteq V_{(i,\alpha)} \) with \( |s| \leq 3 \), and \( c \in C_s \). If there exists \( a \in \text{Sol}(\text{Inst}) \) with \( \text{proj}_s(a) = c \), then for some \( \alpha \)-block \( B \) there exists \( b \in \text{Sol}(\text{Inst}^B_{(i,\alpha)}) \) with \( \text{proj}_s(b) = c \).

Proof. Given \( a \in \text{Sol}(\text{Inst}) \), let \( B = a_{i/\alpha} \) and put \( b = a|_{V_{(i,\alpha)}} \).

\( \square \)

**KEY:** Each \( \text{Inst}^B_{(i,\alpha)} \) can be solved by Gaussian elimination (!), and there are only \( \text{poly}(n) \)-many of them. Hence (using the above Lemma) we can “easily” pre-process \( \text{Inst} \) to enforce the following:

For every prime \( p \), \( (i, \alpha) \in VC_p \), \( s \subseteq V_{(i,\alpha)} \) with \( |s| = 2 \), and \( c \in C_s \), there exists \( b \in \text{Sol}(\text{Inst}_{(i,\alpha)}) \) with \( \text{proj}_s(b) = c \).

Call this condition **2-linear consistency**.
APOLOGY: there is one more technical definition.

It takes 3 slides to explain.
Active 3-ary constraints

Again assume \( \text{Inst} \) is a 3-minimal instance of 3-CSP(\( \mathbb{A} \)).

For any \(|s| = 3\) we always have

\[
\mathcal{C}_s \subseteq \left\{ \mathbf{a} \in \mathbb{A}^s : \text{proj}_t(\mathbf{a}) \in \mathcal{C}_t \text{ for all } t \subseteq s \text{ with } |t| = 2 \right\}.
\]

**Definition.** Call \((s, \mathcal{C}_s)\) passive if \(\mathcal{C}_s = \widehat{\mathcal{C}_s}\), and active if \(\mathcal{C}_s \subsetneq \widehat{\mathcal{C}_s}\).

(Aside: if we start with an instance of 2-CSP(\( \mathbb{A} \)) and enforce 3-minimality, all of the resulting 3-ary constraints will be passive.)
Example – active constraint

Suppose $i \in V$, $\alpha \in \text{Con}(P_i)$, and $\langle 0_{P_i}, \alpha \rangle$ bounds a copy of $M_3$.

Define $H = \{(h_{ij}(a), h_{ik}(a), h_{i\ell}(a)) : a \in P_i\} \subseteq A^{\{j,k,\ell\}}$.

Let $s = \{j, k, \ell\}$. Then $H \subsetneq \widehat{C_s}$. Hence if $C_s = H$, then $(s, C_s)$ is active.
**M₃-induced active constraints**

\[
\text{Con}(P_i) =
\]

- Con (Pᵢ)

**Definition.** Let \( A \) be a finite ternary group and \( \text{Inst} \) a 3-minimal instance of 3-CSP(\( A \)). We say that \( \text{Inst} \) has **M₃-induced active constraints** if for every \( i \in V \), \( \alpha \in \text{Con}(P_i) \), and \( s = \{j, k, \ell\} \subseteq V \) as described on the previous slide, and with \( H = \{(h_{ij}(a), h_{ik}(a), h_{i\ell}(a)) : a \in P_i\} \),

\[
\text{if } \alpha \text{ is elementary } p\text{-abelian for some prime } p, \text{ then } C_s = H.
\]

By adding the constraint \((s, H)\) whenever required, we can easily enforce that \( \text{Inst} \) have **M₃-induced active constraints**.
Pre-processing: Summary

Let $A$ be a finite ternary group. Given an instance of $2$-$\text{CSP}(A)$, we can enforce

- 3-minimality
- Congruence completeness.
- 2-linear consistency
- $M_3$-induced active constraints.

If a contradiction is not found, this “super” pre-processing will produce an equivalent instance of $3$-$\text{CSP}(A)$ which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has $M_3$-induced active constraints;
- has no other active constraints.
Conjecture (Stará Lesná)

Suppose \( A \) is a finite ternary group and \( \text{Inst} \) is an instance of 3-CSP(\( A \)) which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has \( M_3 \)-induced active constraints;
- has no other active constraints.

Then \( \text{Inst} \) has a solution.

If true, we will get the following "easy" algorithm for ternary groups \( A \):

Input: an instance of 2-CSP(\( A \))

"Super" pre-process the instance

If a contradiction is found, return "NO"

Return "Yes"
OK, **maybe** this new algorithm will work for ternary groups . . .

. . . but what does this have to do with Taylor algebras??
Generalizing to Taylor algebras

For general algebras, there is a notion of “abelian congruence.”

If $A$ is finite and has a Taylor term, then every block of an abelian congruence “is” a ternary abelian group (in a natural way).

**Definition.** Let $A$ be a finite algebra with a Taylor term, $\alpha \in \text{Con}(A)$, and $p$ a prime. We say that $\alpha$ is **elementary $p$-abelian** if $\alpha$ is abelian and every $\alpha$-block “is” a ternary abelian group of exponent $p$.

Many facts about finite ternary groups lift to abelian congruences in finite Taylor algebras. For example:

**Proposition**

Let $A$ be a finite algebra with a Taylor term and $\alpha$ a **minimal** congruence. If $\alpha$ is abelian, then $\alpha$ is elementary $p$-abelian for some prime $p$. 
Wild speculation

Let $A$ be any finite, idempotent algebra with a Taylor term.

Let $Inst$ be an instance of 2-CSP($A$).

Just as for ternary groups, we can “super” pre-process $Inst$ to either find a contradiction or produce an equivalent instance of 3-CSP($A$) which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has $M_3$-induced active constraints;
- has no other active constraints.

Problem (Stará Lesná)

For which Taylor varieties is it true that every 3-CSP($A$) instance satisfying the above conditions has a solution? (Could it be all Taylor varieties??)

Thank you!