~~New algebraic insights from the solutions to the dichotomy conjecture~~

What I learned from reading Dmitriy's proof (of the CSP Dichotomy Theorem), Part 5

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Second Algebra Week
Siena
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Motivation

R. Freese, “Subdirectly irreducible algebras in modular varieties,” Springer LNM 1004 (1982).

- Develops a notion of “similarity” between different subdirectly irreducible (SI) algebras with “abelian” monoliths (in CM varieties).

D. Zhuk, ”A proof of CSP Dichotomy Conjecture,” `arXiv:1704:01914` (2017)

- Develops a notion of “bridge” (between certain meet-irreducible congruences of possibly different algebras).
- Results which formally appear to be consequences of a (hypothetical) generalization of Freese’s theory to finite SIs in Taylor varieties.

My goal: to find this generalization.

Plan

I will:

- Carefully formulate and (partly) prove one such generalization.
- State some open problems.

I will not:

- Explain Freese's and Zhuk's original results, or how my results generalize theirs.
- Mention CSP, polymorphisms, minions, etc. (promise!)

I assume you:

- are comfortable with notions from universal algebra, ...
- can tolerate 1.5 hours focused on an algebraic notion ("abelianness") which never arises in the presence of lattice operations, and ...
- are willing to accept ads for tame congruence theory (TCT).

Centrality (via the term condition)

Definition. Let $\alpha, \beta, \delta \in \operatorname{Con} \mathbf{A}$.

<u>α centralizes β modulo δ</u> $\iff$ $\forall$ term $t(\mathbf{x}, \mathbf{y})$, $\forall (a_i, b_i) \in \alpha$, $\forall (c_j, d_j) \in \beta$,

$$t(\mathbf{a}, \mathbf{c}) \stackrel{\delta}{\equiv} t(\mathbf{a}, \mathbf{d}) \iff t(\mathbf{b}, \mathbf{c}) \stackrel{\delta}{\equiv} t(\mathbf{b}, \mathbf{d}).$$

Note: the 2×2 array $\begin{bmatrix} t(\mathbf{a}, \mathbf{c}) & t(\mathbf{a}, \mathbf{d}) \\ t(\mathbf{b}, \mathbf{c}) & t(\mathbf{b}, \mathbf{d}) \end{bmatrix}$ is called an <u>(α, β)-matrix</u>.

More definitions:

- α centralizes β $\iff$ α centralizes β modulo 0.
- $[\alpha, \beta] = 0$ $\iff$ α centralizes β.
- α is abelian $\iff$ $[\alpha, \alpha] = 0$.
- $\mathbf{A}$ is abelian $\iff$ $[1, 1] = 0$.

Fact. Given any algebra $\mathbf{A}$ and any $\beta \in \operatorname{Con}\mathbf{A}$, there exists a unique largest $\alpha \in \operatorname{Con}\mathbf{A}$ which centralizes β.

This α is called the <u>centralizer</u> of β and is denoted $(0 : \beta)$.

Examples:

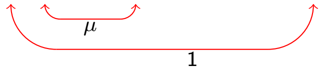
1. In the group $(\mathbb{Z}_4, +)$, if μ is the monolith, then $(0 : \mu) = 1$.

   Proof: $(\mathbb{Z}_4, +)$ is abelian, so $[1, 1] = 0$, so $[1, \mu] = 0$.

2. In the ring $(\mathbb{Z}_4, +, \cdot)$, with μ again the monolith, then $(0 : \mu) = \mu$.

   Proof that $(0 : \mu) \neq 1$:

$$0 \cdot 0 = 0 \cdot 2 \qquad \text{but} \qquad 1 \cdot 0 \neq 1 \cdot 2$$



   Thus 1 does not centralize μ.

Taylor varieties

Definition. A variety $\mathcal{V}$ is Taylor if it satisfies either of the following equivalent conditions:

1. $\mathcal{V}$ satisfies some nontrivial idempotent Maltsev condition ($\equiv$ "satisfies a nontrivial set of idempotent identities" à la Julius).
2. $\mathcal{V}$ has a Taylor term, i.e., a term $t(x_1, \ldots, x_n)$ such that
   - $\mathcal{V} \models t(x, \ldots, x) \approx x$ (t is idempotent)
   - For each $i = 1, \ldots, n$, $\mathcal{V}$ satisfies an identity of the form

$$\underset{\substack{\uparrow \\ i}}{t(\mathrm{vars}, x, \mathrm{vars}')} \approx t(\mathrm{vars}'', \underset{\substack{\uparrow \\ i}}{y}, \mathrm{vars}''')$$

($\equiv$ "satisfies a nontrivial idempotent loop condition" à la Julius).

Tame Congruence Theory

If $\mathcal{V}$ is a locally finite Taylor variety, then:

- $\mathcal{V}$ "omits type 1."
- $\mathcal{V}$ has a "weak near unanimity" (WNU) term.
- $\mathcal{V}$ has a "weak difference term."

Definition.

A <u>weak difference term</u> is a term $d(x, y, z)$ with the following property:

> *For all* $\mathbf{A} \in \mathcal{V}$ *and all* $\alpha \in \operatorname{Con} \mathbf{A}$*, if* α *is abelian then* $d(x, y, z)$ *"is Maltsev" on each* α*-class:*
>
> $$\forall (a, b) \in \alpha, \quad d(a, a, b) = b = d(b, a, a).$$

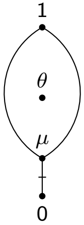
Intuition: $\quad d(x, y, z)|_C = x - y + z \quad$ for $C \in A/\alpha$.

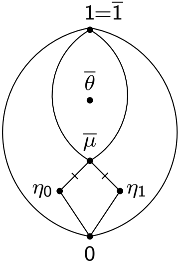
A classic construction

Let **A** be a finite SI with monolith μ in a Taylor variety.

Let $\boldsymbol{\mu}$ be μ considered as a subalgebra of $\mathbf{A}^2$.

Consider Con $\boldsymbol{\mu}$:

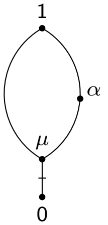
Con **A** =


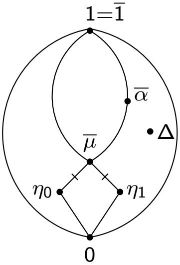
Con $\boldsymbol{\mu}$ =


The kernels of the two projections: η_0 and η_1

For each $\theta \in \text{Con}\,\mathbf{A} \setminus \{0\}$,

$$\overline{\theta} := \{((a_1, a_2), (b_1, b_2)) \in \mu \times \mu : a_1 \overset{\theta}{\equiv} b_1\}.$$

Con $\mathbf{A}$ = 

Con $\boldsymbol{\mu}$ = 

Now assume that μ is abelian.

Fix another congruence α such that $\alpha \geq \mu$ and $[\alpha, \mu] = 0$.

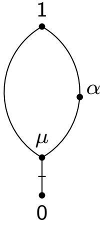
Notation: For each α-class C, let $0_C := \{(a, a) : a \in C\}$.

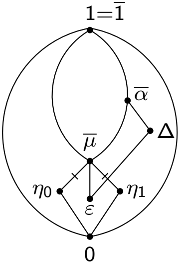
Define

$$\Delta = \Delta_{\alpha,\mu} = \mathrm{Cg}^{\boldsymbol{\mu}}\left(\left\{\ \big((a,a),(b,b)\big) : (a,b) \in \alpha\right\}\right),$$

i.e., the smallest congruence of $\boldsymbol{\mu}$ collapsing each 0_C ($C \in A/\alpha$).

$\Delta = \mathrm{Cg}^{\boldsymbol{\mu}}\left(\left\{\,((a,a),(b,b)) : (a,b) \in \alpha\right\}\right)$, i.e., collapsing 0_C $(C \in A/\alpha)$

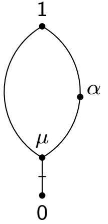
Con $\mathbf{A}$ = 

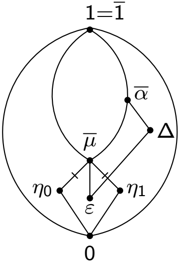
Con $\boldsymbol{\mu}$ = 

Let $\varepsilon := \Delta \wedge \overline{\mu}$. Easy facts:

- $\Delta \leq \overline{\alpha}$.
- $\Delta \vee \eta_0 = \overline{\alpha}$. Proof: $(a_1,a_2) \overset{\eta_0}{\equiv} (a_1,a_1) \overset{\Delta}{\equiv} (b_1,b_1) \overset{\eta_0}{\equiv} (b_1,b_2)$.

  α (linking (a_1,a_2) and (b_1,b_2))
- Similarly, $\varepsilon \vee \eta_0 = \overline{\mu} = \varepsilon \vee \eta_1$.

Con $\mathbf{A}$ = 

Con $\boldsymbol{\mu}$ = 

Fact: in general, $(a, b) \stackrel{\Delta}{\equiv} (a', b')$ iff there exist

$$(a, b) = (a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n) = (a', b')$$

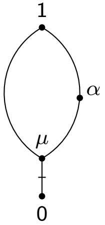
such that each $\begin{bmatrix} a_{i-1} & b_{i-1} \\ a_i & b_i \end{bmatrix}$ is an (α, μ)-matrix.

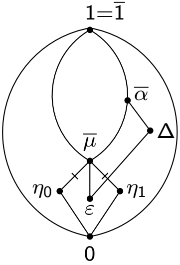
Because $[\alpha, \mu] = 0$, we cannot have $a_{i-1} = b_{i-1}$ and $a_i \neq b_i$.

Hence $\forall a \in A$, $(a, a)/\Delta \subseteq 0_C$ where $C = a/\alpha$.

I.e., 0_C is a Δ-class ($\forall C \in A/\alpha$). This proves $\Delta < \overline{\alpha}$ and $\varepsilon < \overline{\mu}$.

Freese's extra bit

Con $\mathbf{A}$ = 

Con $\boldsymbol{\mu}$ = 

Next goal: to show $\Delta \prec \overline{\alpha}$.

Aside: If Con $\boldsymbol{\mu}$ were modular, this would be easy:

- $(\eta_0, \overline{\mu}) \searrow (0, \eta_1)$.
- Hence $0 \prec \eta_1$. Similarly, $0 \prec \eta_0$ so $\varepsilon \prec \overline{\mu}$ so $\Delta \prec \overline{\alpha}$.

Unfortunately, Con $\boldsymbol{\mu}$ is probably not modular.

Solution: computer-assisted proof!

1. Send email to Keith Kearnes:

Ross Willard May 21, 2019 at 11:24 AM

math question

To: Keith Kearnes

Hi Keith,

I'm preparing Wednesday's talk. There is a small point that I don't know the answer to:

Suppose A is a finite SI algebra in a Taylor variety (you can assume idempotent) with monolith mu.
Assume [1,mu]=0. Let M be the subalgebra of A^2 with universe mu. By the centrality assumption, M as a congruence theta, one of whose blocks is 0_A.

I want to say that M/theta is abelian. I can prove it in difference term varieties. Do you know if it is true in (locally finite) Taylor varieties?

—Ross

2. Wait for answer:

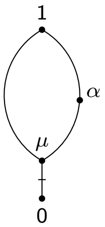
kearnes@colorado.edu <kearnes@Colorado.EDU> May 21, 2019 at 11:56 AM

Re: math question

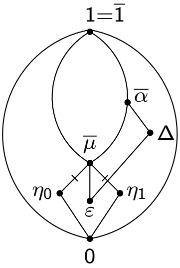
To: Ross Willard

Yes.

Con **A** =



Con $\boldsymbol{\mu}$ =



Recall that **A** belongs to a Taylor variety $\mathcal{V}$.

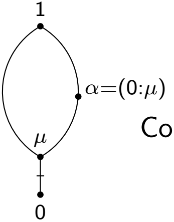
Then by TCT:

- The covers $(\eta_0, \overline{\mu})$ and $(\eta_1, \overline{\mu})$ have type 2.
- So every cover between 0 and $\overline{\mu}$ has type 2 (since $1 \notin \text{typ}\{\mathcal{V}\}$).
- So the interval $I[0, \overline{\mu}]$ is modular.

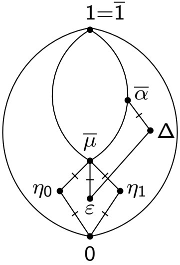
So $\varepsilon \prec \overline{\mu}$ as before.

If $\Delta \not\prec \overline{\alpha}$, then we would get an N_5 with abelian lower cover, impossible (as $1 \notin \text{typ}\{\mathcal{V}\}$). So $\Delta \prec \overline{\alpha}$.

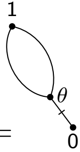
Con $\mathbf{A}$ =



Con $\boldsymbol{\mu}$ =



Con $\mathbf{D}$ =



Now assume that $\alpha = (0 : \mu)$. (the largest such that $[\alpha, \mu] = 0$)

Then one can show Δ is meet-irreducible and $(\Delta : \overline{\alpha}) = \overline{\alpha}$.

Let $\mathbf{D} := \boldsymbol{\mu}/\Delta$ and $\theta = \overline{\alpha}/\Delta$.

$\mathbf{D}$ is SI, its monolith θ is abelian, $(0 : \theta) = \theta$, and $\mathbf{D}/\theta \cong \mathbf{A}/\alpha$. Also,

- $C \in A/\alpha \implies 0_C \in D$.
- $D_o := \{0_C : C \in A/\alpha\}$ is a subuniverse of $\mathbf{D}$. (Because $0_A \leq \boldsymbol{\mu}$.)
- D_o is a transversal for θ.
- The natural map $\nu : \boldsymbol{\mu} \to \mathbf{D}$ satisfies $\nu^{-1}(D_o) = 0_A$.

This proves most of:

Theorem 1

Suppose $\mathbf{A}$ is a finite SI algebra with abelian monolith μ in a Taylor variety.

Let $\alpha = (0 : \mu)$.

There exists an SI algebra $\mathbf{D}$ with abelian monolith θ, a subuniverse $D_o \leq \mathbf{D}$, a surjective homomorphism $h : \boldsymbol{\mu} \twoheadrightarrow \mathbf{D}$, and an isomorphism $h^* : \mathbf{A}/\alpha \cong \mathbf{D}/\theta$ such that:

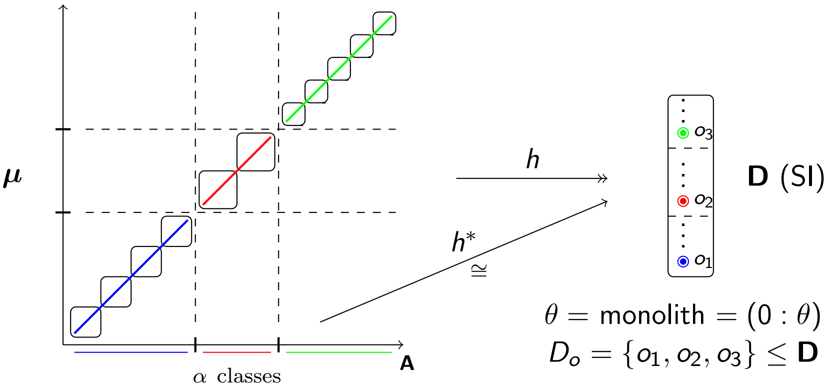
1. $(0 : \theta) = \theta$.
2. D_o is a transversal for θ.
3. $h^{-1}(D_o) = 0_A$.
4. h and h^* are compatible, i.e., $h(a, b)/\theta = h^*(a/\alpha) = h^*(b/\alpha)$.

Moreover, $(\mathbf{D}, D_o)$ is uniquely determined by $\mathbf{A}$ up to isomorphism.

Now forget how we constructed $\mathbf{D}$ (via Con $\boldsymbol{\mu}$, Δ). Focus on this:

Given $\mathbf{A}$ finite SI in Taylor variety, abel. monolith μ, $(0:\mu) = \alpha$,

$\exists$ essentially unique $(\mathbf{D}, D_o)$ (and $h : \boldsymbol{\mu} \twoheadrightarrow \mathbf{D}$ and $h^* : \mathbf{A}/\alpha \cong \mathbf{D}/\theta$) s.t.



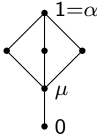
θ = monolith = $(0:\theta)$

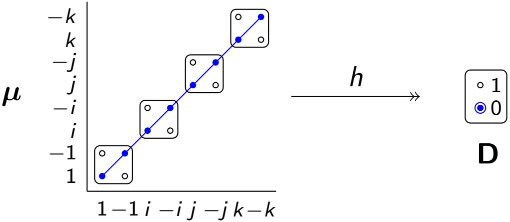
$D_o = \{o_1, o_2, o_3\} \leq \mathbf{D}$

Example

Let $\mathbf{A}$ = the quaternion group $\mathbf{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

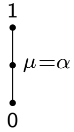
- $\mathbf{Q}_8$ is SI, monolith μ is abelian.
- $(0 : \mu) = 1$.
- μ has classes $\{\pm 1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$.
- Theorem 1 is witnessed by the group $\mathbf{D} = (\mathbb{Z}_2, +)$ and $\{o\} = \{0\}$: $h : \boldsymbol{\mu} \twoheadrightarrow \mathbf{D}$ sends all $(x, x) \mapsto 0$ and all $(x, -x) \mapsto 1$.





Example

Let $\mathbf{A}$ = the symmetric group $\mathbf{S}_3$.

$\mathrm{Con}\,\mathbf{S}_3 =$ 

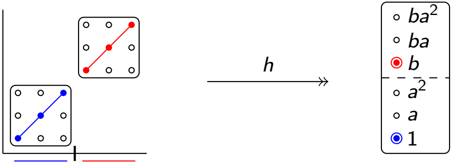
Observe:

- $(0 : \mu) = \mu$.
- The subgroup $\langle b \rangle = \{1, b\}$ is a transversal for $(0 : \mu) \ \ (= \mu)$.

Claim: $(\mathbf{D}, D_o) := (\mathbf{S}_3, \langle b \rangle)$ witnesses Theorem 1 for $\mathbf{S}_3$.

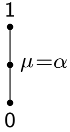
Proof: define $h : \boldsymbol{\mu} \twoheadrightarrow \mathbf{S}_3$ by

$$h(x,y) = \begin{cases} xy^{-1}1 & \text{if } x \overset{\mu}{\equiv} y \overset{\mu}{\equiv} 1 \\ xy^{-1}b & \text{if } x \overset{\mu}{\equiv} y \overset{\mu}{\equiv} b. \end{cases}$$



Example

Let **A** = the ring $(\mathbb{Z}_4, +, \cdot)$.

Con $\mathbb{Z}_4$ = 

Again $(0 : \mu) = \mu$.

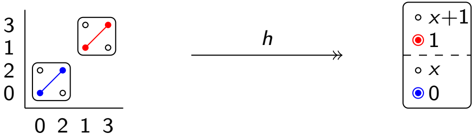
But $\mathbb{Z}_4$ does not have a subring which is a transversal for μ.

If you carry out the construction of $\boldsymbol{\mu}/\Delta$, you get the 4-element ring

$$\mathbf{D} = \mathbb{Z}_2[x]/\langle x^2\rangle = \{0, 1, x, x{+}1\}.$$

The subring $\mathbb{Z}_2$ is a transversal for the monolith of **D**.

We can define $h : \boldsymbol{\mu} \twoheadrightarrow \mathbf{D}$ by



$(\mathbf{D}, \mathbb{Z}_2)$ witnesses Theorem 1 for $(\mathbb{Z}_4, +, \cdot)$.

Similarity

The output of Theorem 1 captures some information about a finite SI with abelian monolith (in a Taylor variety).

Definition

Suppose $\mathbf{A}, \mathbf{B}$ are finite SIs with abelian monoliths in a Taylor variety.

We say $\mathbf{A}, \mathbf{B}$ are <u>similar</u> and write $\mathbf{A} \sim \mathbf{B}$ if they have the same (up to $\cong$) output $(\mathbf{D}, D_o)$ from Theorem 1.

Examples:

1. $\mathbf{Q}_8 \sim (\mathbb{Z}_4, +)$. $((\mathbb{Z}_2, +), \{0\})$ witnesses Theorem 1 for both.
2. $\mathbf{S}_3 \sim (\mathbb{Z}_9, +)$? No. $(0 : \mu_{\mathbf{S}_3}) = \mu_{\mathbf{S}_3}$ while $(0 : \mu_{\mathbb{Z}_9}) = 1$.

Note: $\sim$ extends Freese's notion of similarity in the CM case.

Who cares?

I've argued elsewhere (parts 1–3 at BLAST 2019) that $\sim$ exactly captures which finite SIs (with abelian monoliths in Taylor varieties) can jointly encode linear equations in their monolith classes.

E.g., $\mathbf{Q}_8$ and $(\mathbb{Z}_4, +)$ can jointly support (subdirect) relations encoding linear equations over $\mathbb{Z}_2$.

This is somehow relevant to CSP.

NOW FOR SOMETHING

COMPLETELY DIFFERENT

Bridges

Definition (≈ Zhuk)

Let $\mathbf{A}, \mathbf{B}$ be finite SIs with monoliths $\mu_{\mathbf{A}}, \mu_{\mathbf{B}}$.

A <u>bridge from $\mathbf{A}$ to $\mathbf{B}$</u> is a subuniverse $R \leq \mathbf{A} \times \mathbf{A} \times \mathbf{B} \times \mathbf{B}$ such that

(1) $\mathrm{proj}_{1,2}(R) = \mu_{\mathbf{A}}$ and $\mathrm{proj}_{3,4}(R) = \mu_{\mathbf{B}}$

(2) $(a_1, a_2, b_1, b_2) \in R$ implies ($a_1 = a_2$ iff $b_1 = b_2$).

Variations:

- A <u>weak bridge</u> is defined as above except (1) is weakened to

  $(1)_w$ $\mathrm{proj}_{1,2}(R) \supset 0_A$ and $\mathrm{proj}_{3,4}(R) \supset 0_B$.

- A bridge is <u>proper</u> if it additionally satisfies

  (3) $(a_1, a_2, b_1, b_2) \in R$ implies $(a_i, a_i, b_i, b_i) \in R$ for $i = 1, 2$.

Example 1

Suppose $\mathbf{A}, \mathbf{B}$ are finite isomorphic SIs and $f : \mathbf{A} \cong \mathbf{B}$.

Define

$$R = \{(a_1, a_2, f(a_1), f(a_2)) \,:\, (a_1, a_2) \in \mu_{\mathbf{A}}\}.$$

R is a proper bridge from $\mathbf{A}$ to $\mathbf{B}$.

Proof. $R \leq \mathbf{A} \times \mathbf{A} \times \mathbf{B} \times \mathbf{B}$ obviously.

(1) $\mathrm{proj}_{1,2}(R) = \mu_{\mathbf{A}}$ obviously. $\mathrm{proj}_{3,4}(R) = \mu_{\mathbf{B}}$ because f is an $\cong$.

(2) $a_1 = a_2$ iff $f(a_1) = f(a_2)$ obviously.

(3) $(a_1, a_2) \in \mu_{\mathbf{A}}$ implies $(a_1, a_1), (a_2, a_2) \in \mu_{\mathbf{A}}$.
So $(a_1, a_2, b_1, b_2) \in R$ implies $(a_1, a_1, b_1, b_1), (a_2, a_2, b_2, b_2) \in R$.

Example 2

Suppose $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are finite SIs in a Taylor variety with $n \geq 3$.

Also assume that 0_{A_i} is meet-irreducible in the lattice of subuniverses of $(\mathbf{A}_i)^2$ for each i.

Let $\rho \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ be an invariant relation satisfying:

- ρ is "functional at every variable."

  (If $\mathbf{a}, \mathbf{b} \in \rho$ agree at $n-1$ coordinates, then $\mathbf{a} = \mathbf{b}$.)

- ρ is not defined by its projections onto $n-1$ coordinates.

  ($\exists \mathbf{c} \notin \rho$ such that $\mathrm{proj}_{[n]\setminus\{i\}}(\mathbf{c}) \in \mathrm{proj}_{[n]\setminus\{i\}}(\rho)$ for all $i \in [n]$.)

Define

$$R = \{(a, a', b, b') \in A_1 \times A_1 \times A_n \times A_n : \exists \mathbf{e} \in A_2 \times \cdots \times A_{n-1} \text{ s.t. } (a, \mathbf{e}, b), (a', \mathbf{e}, b') \in \rho\}.$$

Then $R \cap (\mu_{\mathbf{A}_1} \times \mu_{\mathbf{A}_n})$ is a proper bridge from $\mathbf{A}_1$ to $\mathbf{A}_n$.

$$R = \{(a,a',b,b') \in A_1 \times A_1 \times A_n \times A_n : \exists \mathbf{e} \in A_2 \times \cdots \times A_{n-1} \text{ s.t. } (a,\mathbf{e},b),(a',\mathbf{e},b') \in \rho\}.$$

For now, I will just prove that R is a weak bridge.

All but the first bridge property are easy to check.

(2) If $(a,a,b,b') \in R$ then $(a,\mathbf{e},b),(a,\mathbf{e},b') \in \rho$, so $b = b'$ since ρ is functional at coordinate n.

(3) If $(a,a',b,b') \in R$ witnessed by $\mathbf{e}$, then $\mathbf{e}$ also witnesses $(a,a,b,b),(a',a',b',b') \in R$.

$(1)_w$ Obviously $\text{proj}_{1,2}(R) \supseteq 0_{A_1}$.

Recall the tuple $\mathbf{c} \notin \rho$ such that $\text{proj}_{[n]\setminus\{i\}}(\mathbf{c}) \in \text{proj}_{[n]\setminus\{i\}}(\rho)$ for all i.

Write $\mathbf{c} = (a,\mathbf{e},b')$. Then for some a', b,

$$\begin{aligned} (a,\mathbf{e},b) &\in \rho \\ (a',\mathbf{e},b') &\in \rho. \end{aligned}$$

Thus $(a,a',b,b') \in R$ and obviously $a \neq a'$. So $\text{proj}_{1,2}(R) \supset 0_{A_1}$.

Inverse and Composition

Suppose R is a bridge from $\mathbf{A}$ to $\mathbf{B}$, and S is a bridge from $\mathbf{B}$ to $\mathbf{C}$.

$$\begin{array}{ccccc} & x_1 & \text{———} & y_1 & \\ \mu_{\mathbf{A}} & \Big| & R & \Big| & \mu_{\mathbf{B}} \\ & x_2 & \text{———} & y_2 & \end{array} \qquad \begin{array}{ccccc} & x_1 & \text{———} & y_1 & \\ \mu_{\mathbf{B}} & \Big| & S & \Big| & \mu_{\mathbf{C}} \\ & x_2 & \text{———} & y_2 & \end{array}$$

The following gadgets define bridges from $\mathbf{B}$ to $\mathbf{A}$ and from $\mathbf{A}$ to $\mathbf{C}$:

$$\begin{array}{ccccc} & y_1 & \text{———} & x_1 & \\ \mu_{\mathbf{A}} & \Big| & R & \Big| & \mu_{\mathbf{B}} \\ & y_2 & \text{———} & x_2 & \end{array} \qquad \begin{array}{ccccccc} & x_1 & \text{———} & \exists & \text{———} & y_1 & \\ \mu_{\mathbf{A}} & \Big| & R & \Big| & S & \Big| & \mu_{\mathbf{C}} \\ & x_2 & \text{———} & \exists & \text{———} & y_2 & \end{array}$$

Call these bridges R^{-1} and $R \circ S$.

Note: If R and S are proper, so are R^{-1} and $R \circ S$.

Lemma

Suppose $\mathbf{A}, \mathbf{B}$ are finite SIs and R is a bridge from $\mathbf{A}$ to $\mathbf{B}$. Let $\mu_{\mathbf{A}}$ be the monolith of $\mathbf{A}$.

If $(x,x,u,u), (y,y,u,u) \in R$ then $(x,y) \in (0 : \mu_{\mathbf{A}})$.

Proof sketch. Letting

$$R^* = (R \circ R^{-1}) \circ (R \circ R^{-1}) \circ \cdots \circ (R \circ R^{-1}) \circ \cdots$$

we get a bridge from $\mathbf{A}$ to $\mathbf{A}$ containing $R \circ R^{-1}$ and satisfying

$$\theta := \{(a,b) \in A^2 \,:\, (a,a,b,b) \in R^*\} \;\in\; \operatorname{Con} \mathbf{A}.$$

In particular, $(x,y) \in \theta$. Moreover,

$$(c,d,c,d) \in R^* \quad \text{for all} \quad (c,d) \in \mu_{\mathbf{A}}.$$

Thus for any term t and $(a_i, b_i) \in \theta$ and $(c_j, d_j) \in \mu_{\mathbf{A}}$,

$$(t(\mathbf{a},\mathbf{c}), t(\mathbf{a},\mathbf{d}), t(\mathbf{b},\mathbf{c}), t(\mathbf{b},\mathbf{d})) \in R^*,$$

which with a bridge property proves $[\theta, \mu_{\mathbf{A}}] = 0$ and so $\theta \le (0 : \mu_{\mathbf{A}})$. □

Now we can finish Example 2:

- $\mathbf{A}_1, \ldots, \mathbf{A}_n$ finite SIs in a Taylor variety, $n \geq 3$.
- $\rho \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$
- $R = \{(a, a', b, b') \in A_1 \times A_1 \times A_n \times A_n : \exists \mathbf{e} \in A_2 \times \cdots \times A_{n-1} \text{ s.t. } (a, \mathbf{e}, b), (a', \mathbf{e}, b') \in \rho\}.$

We've shown that R is a weak bridge from $\mathbf{A}_1$ to $\mathbf{A}_n$. Also observe that

$$(a, b) \in \mathrm{proj}_{1,n}(\rho) \implies (a, a, b, b) \in R.$$

Symmetrically, $\exists$ weak bridge S from $\mathbf{A}_1$ to $\mathbf{A}_2$ with

$$(a, c) \in \mathrm{proj}_{1,2}(\rho) \implies (a, a, c, c) \in S.$$

Recall the tuples $(a, \mathbf{e}, b), (a', \mathbf{e}, b') \in \rho$ with $a \neq a'$.

Thus $(a, e_1), (a', e_1) \in \mathrm{proj}_{1,2}(\rho)$.

So $(a, a, e_1, e_1), (a', a', e_1, e_1) \in S$.

S a weak bridge from $\mathbf{A}_1$ to $\mathbf{A}_2$.

$(a, a, e_1, e_1), (a', a', e_1, e_1) \in S$ with $a \neq a'$.

Thus by the Lemma, $(a, a') \in (0 : \mu_{\mathbf{A}_1})$. **(I cheated here.**[1]**)**

So $(0 : \mu_{\mathbf{A}_1}) \neq 0$.

So $(0 : \mu_{\mathbf{A}_1}) \geq \mu_{\mathbf{A}_1}$.

I.e., $\mu_{\mathbf{A}_1}$ is abelian.

Recall that $\mathbf{A}_1$ has a weak difference term.

It follows that $\mu_{\mathbf{A}_1}$ is the unique smallest subuniverse of $(\mathbf{A}_1)^2$ properly containing 0_{A_1}.

Since $\mathrm{proj}_{1,2}(R) \supset 0_{A_1}$ (already proved), get $\mathrm{proj}_{1,2}(R) \supseteq \mu_{\mathbf{A}_1}$.

Similarly, $\mathrm{proj}_{3,4}(R) \supseteq \mu_{\mathbf{A}_n}$.

So $R \cap (\mu_{\mathbf{A}_1} \times \mu_{\mathbf{A}_n}) \leq_{sd} \mu_{\mathbf{A}_1} \times \mu_{\mathbf{A}_n}$ which finishes the proof of property (1).

[1] I proved the Lemma only for bridges. Using TCT, can prove it for weak bridges.

Bridges were an essential tool in Zhuk's proof of the CSP Dichotomy Theorem. (See parts 1–3 from BLAST 2019).

But how are they connected to similarity?

Theorem 2

Let $\mathbf{A}, \mathbf{B}$ be finite SIs with abelian monoliths $\mu_{\mathbf{A}}, \mu_{\mathbf{B}}$ in a Taylor variety.

$\mathbf{A} \sim \mathbf{B}$ iff there exists a proper bridge from $\mathbf{A}$ to $\mathbf{B}$.

OMG!! (Proper) bridges = similarity!

Theorem 2

Let $\mathbf{A}, \mathbf{B}$ be finite SIs with abelian monoliths $\mu_{\mathbf{A}}, \mu_{\mathbf{B}}$ in a Taylor variety.

$\mathbf{A} \sim \mathbf{B}$ iff there exists a proper bridge from $\mathbf{A}$ to $\mathbf{B}$.

Proof sketch.

($\Rightarrow$) Let $(\mathbf{D}, D_o)$, $h_{\mathbf{A}} : \mu_{\mathbf{A}} \twoheadrightarrow \mathbf{D}$ and $h_{\mathbf{B}} : \mu_{\mathbf{B}} \twoheadrightarrow \mathbf{D}$ witness $\mathbf{A} \sim \mathbf{B}$.

Define

$$R = \{(a, b, r, s) \in \mu_{\mathbf{A}} \times \mu_{\mathbf{B}} \,:\, h_{\mathbf{A}}(a, b) = h_{\mathbf{B}}(r, s)\}.$$

It is a proper bridge.

($\Leftarrow$) Let R be a proper bridge from $\mathbf{A}$ to $\mathbf{B}$.

Using $d(x, y, z)$, show that for all $(a, b, r, s), (a', b', r', s') \in R$,

$$(a, b) \overset{\Delta_{\mathbf{A}}}{\equiv} (a', b') \iff (r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (r', s').$$

Thus R defines an isomorphism $\boldsymbol{\mu}/\Delta_{\mathbf{A}} \cong \boldsymbol{\mu}/\Delta_{\mathbf{B}}$. □

Full proof.

Let $\alpha_{\mathbf{A}} = (0_A : \mu_{\mathbf{A}})$ and $\alpha_{\mathbf{B}} = (0_B : \mu_{\mathbf{B}})$.

($\Rightarrow$) Let $(\mathbf{D}, D_o)$, $h_{\mathbf{A}} : \mu_{\mathbf{A}} \twoheadrightarrow \mathbf{D}$ and $h_{\mathbf{B}} : \mu_{\mathbf{B}} \twoheadrightarrow \mathbf{D}$ witness $\mathbf{A} \sim \mathbf{B}$.

Define

$$R = \{(a, b, r, s) \in \mu_{\mathbf{A}} \times \mu_{\mathbf{B}} \,:\, h_{\mathbf{A}}(a, b) = h_{\mathbf{B}}(r, s)\}.$$

Clearly

- $\mathrm{proj}_{1,2}(R) = \mu_{\mathbf{A}}$ and $\mathrm{proj}_{3,4}(R) = \mu_{\mathbf{B}}$.
- $(a, a, r, s) \in R \implies h_{\mathbf{B}}(r, s) = h_{\mathbf{A}}(a, a) \in D_o \implies r = s$.
  (And symmetrically.)

So R is a bridge.

- To prove proper, assume $(a, b, r, s) \in R$, so $h_{\mathbf{A}}(a, b) = h_{\mathbf{B}}(r, s) =: e$.

  By compatibility,

  $$e/\theta = h_{\mathbf{A}}(a, b)/\theta = h^*_{\mathbf{A}}(a/\alpha_{\mathbf{A}}) = h_{\mathbf{A}}(a, a)/\theta,$$

  proving $h_{\mathbf{A}}(a, a) \overset{\theta}{\equiv} e$. Similarly, $h_{\mathbf{B}}(r, r) \overset{\theta}{\equiv} e$.

  Thus $h_{\mathbf{A}}(a, a)$ and $h_{\mathbf{B}}(r, r)$ are θ-related <u>and</u> in D_o.

  Thus $h_{\mathbf{A}}(a, a) = h_{\mathbf{B}}(r, r)$. ($D_o$ is a transversal for θ.)

  Thus $(a, a, r, r) \in R$.

  $(b, b, s, s) \in R$ is proved similarly.

Conclusion: R is a proper bridge from **A** to **B**.

($\Leftarrow$) Assume there exists a proper bridge R from $\mathbf{A}$ to $\mathbf{B}$.

Define $\Delta_{\mathbf{A}} \in \operatorname{Con} \boldsymbol{\mu}_{\mathbf{A}}$ and let $T_{\mathbf{A}}$ denote $\{0_C : C \in A/\alpha_{\mathbf{A}}\} \leq \boldsymbol{\mu}_{\mathbf{A}}/\Delta_{\mathbf{A}}$.

Recall that $(\boldsymbol{\mu}_{\mathbf{A}}/\Delta_{\mathbf{A}}, T_{\mathbf{A}})$ witnesses Theorem 1 for $\mathbf{A}$.

Similarly, $(\boldsymbol{\mu}_{\mathbf{B}}/\Delta_{\mathbf{B}}, T_{\mathbf{B}})$ witnesses Theorem 1 for $\mathbf{B}$.

Thus it will suffice to find $h : \boldsymbol{\mu}_{\mathbf{A}}/\Delta_{\mathbf{A}} \cong \boldsymbol{\mu}_{\mathbf{B}}/\Delta_{\mathbf{B}}$ taking $T_{\mathbf{A}}$ to $T_{\mathbf{B}}$.

Main claim: for all $(a, b, r, s), (a', b', r', s') \in R$,

$$(a, b) \overset{\Delta_{\mathbf{A}}}{\equiv} (a', b') \iff (r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (r', s').$$

This claim will suffice, for then we can define

$$h((a, b)/\Delta_{\mathbf{A}}) = (r, s)/\Delta_{\mathbf{B}}, \qquad \text{any } (a, b, r, s) \in R.$$

Before proving the Main claim, we need the following

Lemma

Suppose $\mathbf{A}$ is SI with abelian monolith μ. Let $\alpha = (0 : \mu)$. Define

$$\begin{aligned} R_\alpha &= \mathrm{Sg}^{\mathbf{A}^4}\left(\left\{(a,a,b,b) : (a,b) \in \alpha\right\} \cup \left\{(c,d,c,d) : (c,d) \in \mu\right\}\right) \\ &= \left\{\left(t(\mathbf{a},\mathbf{c}), t(\mathbf{a},\mathbf{d}), t(\mathbf{b},\mathbf{c}), t(\mathbf{b},\mathbf{d})\right) : \right. \\ &\qquad\qquad \left. t \text{ a term}, (a_i,b_i) \in \alpha, (c_j,d_j) \in \mu\right\} \end{aligned}$$

(i.e., the set of all 4-tuples obtained from the rows of (α,μ)-matrices).

R_α is a proper bridge from $\mathbf{A}$ to $\mathbf{B}$.

Proof.

- Clearly $\mathrm{proj}_{1,2}(R_\alpha) = \mathrm{proj}_{3,4}(R_\alpha) = \mu$.
- Clearly $t(\mathbf{a},\mathbf{c}) = t(\mathbf{a},\mathbf{d})$ iff $t(\mathbf{b},\mathbf{c}) = t(\mathbf{b},\mathbf{d})$ since $[\alpha,\mu] = 0$.
- Can check R_α is proper (by setting $\mathbf{c} = \mathbf{d}$). □

Main claim: for all $(a, b, r, s), (a', b', r', s') \in R$,

$$(a, b) \overset{\Delta_{\mathbf{A}}}{\equiv} (a', b') \iff (r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (r', s').$$

Proof of Main claim.

Recall the proper bridges $R_{\alpha_{\mathbf{A}}}$ (from **A** to **A**) and $R_{\alpha_{\mathbf{B}}}$ (from **B** to **B**).

Let $\mathrm{tr}(R) = \{(a, r) \in A \times B : (a, a, r, r) \in R\}$.

Let $R' = R_{\alpha_{\mathbf{A}}} \circ R \circ R_{\alpha_{\mathbf{B}}}$.

R' is also a proper bridge from **A** to **B**, $R \subseteq R'$, and the set

$$\mathrm{tr}(R') := \{(a, r) \in A \times B : (a, a, r, r) \in R'\}$$

satisfies $\mathrm{tr}(R') = \alpha_{\mathbf{A}} \circ \mathrm{tr}(R') \circ \alpha_{\mathbf{B}}$.

It suffices to prove the Main claim for R'.

Thus WLOG we can assume $R' = R$.

Assume $(a, b, r, s), (a', b', r', s') \in R$ and $(a, b) \overset{\Delta_{\mathbf{A}}}{\equiv} (a', b')$.

Recall that this means there exist

$$(a, b) = (a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n) = (a', b')$$

such that each $\begin{bmatrix} a_{i-1} & b_{i-1} \\ a_i & b_i \end{bmatrix}$ is an $(\alpha_{\mathbf{A}}, \mu_{\mathbf{A}})$-matrix.

Extend each (a_i, b_i) $(i \neq 0, n)$ to $(a_i, b_i, r_i, s_i) \in R$. Thus

$$\begin{array}{rcl} (a, b, r, s) & \in & R \\ (a_1, b_1, r_1, s_1) & \in & R \\ (a_2, b_2, r_2, s_2) & \in & R \\ & \vdots & \\ (a', b', r', s') & \in & R. \end{array}$$

Suffices to prove $(r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (r_1, s_1)$. Reset $(a', b', r', s') \leftarrow (a_1, b_1, r_1, s_1)$.

So assume $(a,b,r,s),(a',b',r',s') \in R$ and

$$\begin{bmatrix} a & b \\ a' & b' \end{bmatrix} = \begin{bmatrix} t(\mathbf{c},\mathbf{f}) & t(\mathbf{c},\mathbf{g}) \\ t(\mathbf{d},\mathbf{f}) & t(\mathbf{d},\mathbf{g}) \end{bmatrix}, \qquad (c_i,d_i) \in \alpha_{\mathbf{A}},\ (f_j,g_j) \in \mu_{\mathbf{A}}.$$

WTP: $(r,s) \stackrel{\Delta_{\mathbf{B}}}{\equiv} (r',s')$. For each i pick $x_i \in B$ with $(c_i,x_i) \in \mathrm{tr}(R)$.

$c_i \stackrel{\alpha_{\mathbf{A}}}{\equiv} d_i$ and (assumption on R) $\implies$ $(d_i,x_i) \in \mathrm{tr}(R)$.

Extend each (f_j,g_j) to $(f_j,g_j,y_j,z_j) \in R$.

Consider the matrices

$$\cdots \left[\begin{array}{cc|cc} c_i & c_i & x_i & x_i \\ d_i & d_i & x_i & x_i \end{array}\right] \cdots \quad \text{and} \quad \cdots \left[\begin{array}{cc|cc} f_j & g_j & y_j & z_j \\ f_j & g_j & y_j & z_j \end{array}\right] \cdots$$

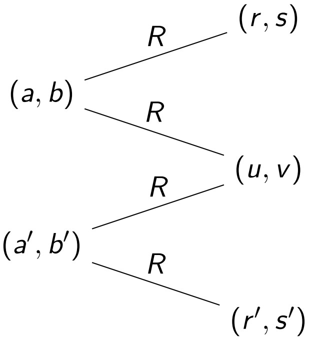
Note that each row of each matrix is in R.

Applying t coordinatewise gives

$$\left[\begin{array}{cc|cc} t(\mathbf{c},\mathbf{f}) & t(\mathbf{c},\mathbf{g}) & t(\mathbf{x},\mathbf{y}) & t(\mathbf{x},\mathbf{z}) \\ t(\mathbf{d},\mathbf{f}) & t(\mathbf{d},\mathbf{g}) & t(\mathbf{x},\mathbf{y}) & t(\mathbf{x},\mathbf{z}) \end{array}\right] = \left[\begin{array}{cc|cc} a & b & u & v \\ a' & b' & u & v \end{array}\right]$$

(for some u,v), and the rows are in R.

So we have



WTP: $(r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (r', s')$.

Suffices to prove $(r, s) \overset{\Delta_{\mathbf{B}}}{\equiv} (u, v)$.

Thus assume

$$\begin{array}{llllll}(a, & b, & r, & s) & \in & R \\ (a, & b, & u, & v) & \in & R.\end{array}$$

WTP: $(r,s) \overset{\Delta_{\mathbf{B}}}{\equiv} (u,v)$.

We have

$$\begin{array}{llllll}(a, & a, & r, & r) & \in & R \\ (a, & a, & u, & u) & \in & R\end{array} \qquad \text{as } R \text{ is proper.}$$

Then by the Lemma, $(r,u) \in \alpha_{\mathbf{B}}$ so also $(s,u),(s,v) \in \alpha_{\mathbf{B}}$.

Consider some $(\alpha_{\mathbf{B}}, \mu_{\mathbf{B}})$-matrices:

$$\begin{bmatrix} r & s \\ r & s \end{bmatrix}, \begin{bmatrix} r & r \\ u & u \end{bmatrix}, \begin{bmatrix} r & r \\ u & u \end{bmatrix} \quad \overset{d}{\Longrightarrow} \quad \begin{bmatrix} r & s \\ d(r,u,u) & d(s,u,u) \end{bmatrix}$$

and

$$\begin{bmatrix} s & s \\ v & v \end{bmatrix}, \begin{bmatrix} v & v \\ v & v \end{bmatrix}, \begin{bmatrix} u & v \\ u & v \end{bmatrix} \quad \overset{d}{\Longrightarrow} \quad \begin{bmatrix} d(s,v,u) & d(s,v,v) \\ u & v \end{bmatrix}.$$

Suffices to prove $d(r,u,u) = d(s,v,u)$ and $d(s,u,u) = d(s,v,v)$.

Summary:

$$\begin{array}{ccccccc} (a, & b, & r, & s) & \in & R \\ (a, & b, & u, & v) & \in & R \\ (a, & a, & u, & u) & \in & R, \end{array}$$

and also $(s, u) \in \alpha_{\mathbf{B}}$.

WTP: $\boxed{d(r, u, u) = d(s, v, u)}$ and $\boxed{d(s, u, u) = d(s, v, v)}$.

2nd is easy:

$$d(\underline{u}, u, u) = d(\underline{u}, v, v) \implies d(\underline{s}, u, u) = d(\underline{s}, v, v) \quad \text{as } [\alpha_{\mathbf{B}}, \mu_{\mathbf{B}}] = 0.$$

For the 1st, apply d to the above three tuples in R to get

$$(a, \quad a, \quad d(r, u, u), \quad d(s, v, u)) \in R$$

By bridge properties, the last two entries are equal. □

Conclusion

There is a notion of "similarity" between finite SIs with abelian monoliths in Taylor varieties (extending the classical notion and Zhuk's bridges).

Problems:

1. Do either of the following conditions characterize $\mathbf{A} \sim \mathbf{B}$?
   (a) $\exists \mathbf{C} \leq_{sd} \mathbf{A} \times \mathbf{B}$, $\exists \delta \prec \gamma \in \operatorname{Con} \mathbf{C}$ such that $\eta_0^*/\eta_0 \searrow \gamma/\delta \nearrow \eta_1^*/\eta_1$.
   (b) $\exists \mathbf{C} \in \mathsf{HSP}(\mathbf{A}, \mathbf{B})$, $\exists R \leq_{sd} \mathbf{A} \times \mathbf{B} \times \mathbf{C}$ s.t. R is "critical and fork-free."
2. Which theorems about similarity in the CM case extend to finite SIs in Taylor varieties?
   (a) E.g. (Freese & McKenzie) If $\mathbf{A}$ is a finite algebra in a CM variety, $\mathbf{B} \in \mathsf{HSP}(\mathbf{A})$, and $\mathbf{B}$ is SI, then $\mathbf{B}$ is similar to an SI algebra in $\mathsf{HS}(\mathbf{A})$.
3. How can similarity be usefully defined between finite SIs with nonabelian monolith in Taylor varieties?
   (a) E.g., extending the above theorem of Freese & McKenzie.

4. Does Theorem 1 extend to infinite SIs in varieties with a weak difference term?

5. Do there exist finite SIs $\mathbf{A}, \mathbf{B}$ with abelian monoliths in a Taylor variety which are connected by a bridge, but not by any proper bridge?

Thank you!