Near unanimity constraints have bounded pathwidth duality

Libor Barto  Marcin Kozik  Ross Willard*
Charles U.  Jagiellonian U.  U. Waterloo

LICS 2012
Dubrovnik, Croatia
June 27, 2012
This talk is about variations of 3-SAT...

For example,

- 2-SAT: \( x \lor y = 1, \quad \overline{x} \lor y = 1, \quad \overline{x} \lor \overline{y} = 1 \)
- Horn (3-)SAT: \( \overline{x} \lor \overline{y} \lor z = 1, \quad \overline{x} \lor \overline{y} \lor \overline{z} = 1, \quad x = 1 \)
- 3-XORSAT: \( x \oplus y \oplus z = 0, \quad x \oplus y \oplus z = 1 \)

Unlike 3-SAT, these variations are all tractable (in P).

However, they have different complexities:

- 2-SAT is NL-complete.
- Horn SAT is P-complete.
- 3-XORSAT is \( \oplus L \)-complete.
These variations are called constraint satisfaction problems (CSPs).

**Definition**

Let $\mathcal{R}$ be a finite set of nonempty boolean relations (i.e., on $\{0, 1\}$).

$\text{CSP}(\mathcal{R})$ is the variant of 3-SAT in which clauses are replaced by $\mathcal{R}$-constraints:

- i.e., assertions that tuples of variables belong to specific relations in $\mathcal{R}$.

A celebrated result:

**Boolean Dichotomy Theorem (Schaefer, 1978)**

*For any finite set $\mathcal{R}$ of boolean relations, $\text{CSP}(\mathcal{R})$ is NP-complete or in P.*
The cases in P were further delineated by Allender et al, who showed that there are precisely 5 possible complexities.

They also characterized the sets $R$ of each complexity. For example:

**Corollary of (Allender, Bauland, Immerman, Schnoor, Vollmer, 2009)**

(Assume $\oplus L \not\subseteq NL$.) Let $R$ be a finite set of nonempty boolean relations, and suppose $R$ is “nontrivial.” The following are equivalent:

1. $\text{CSP}(R) \in NL$.
2. $R$ does not “interpret” Horn SAT or 3-XORSAT.
3. $\exists$ function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ (for some $n \geq 1$) which
   - preserves each relation in $R$
   - satisfies
     
     \[
     f(a, \ldots , a, b, a, \ldots , a) = a \\
     \uparrow
     \\
     i
     \]
     \[
     \forall a, b \in \{0, 1\} \\
     \forall 1 \leq i \leq n
     \]
   - “near unanimity”

"polymorphism"
This paper is a contribution to ongoing efforts to extend the Schaefer Dichotomy and its refined version to larger (non-boolean) domains...

...specifically, the following conjectured extension of the Allender et al corollary:

**Conjecture (Larose, Tesson, 2009)**

(Assume Mod\(p\)L \(\not\subset\) NL for all primes \(p\).)

Let \(R\) be a finite set of nonempty relations on a finite domain, and suppose \(R\) is “not reducible” (i.e., core). The following are equivalent:

1. \(\text{CSP}(R) \in \text{NL}\).
2. \(R\) does not “interpret” Horn SAT or 3-LinEq(\(F\)) for any finite field \(F\).
Conjecture (restated)

$(\text{Mod}_p L \not\subseteq NL)$ Core $\mathcal{R}$ over general domains, the following are equivalent:

1. $\text{CSP}(\mathcal{R}) \in NL$.
2. $\mathcal{R}$ does not “interpret” Horn SAT or $3\text{-LinEq}(F)$ for any finite field $F$.

Remarks

- $(1) \Rightarrow (2)$ is “obvious.”
  - It follows from the complexity assumption, the notion of “interpret,” and the known complexities of Horn SAT and $3\text{-LinEq}(F)$.

- Universal algebraists know a “polymorphism characterization” of $(2)$ over general domains. It is related to, but strictly weaker than, $\mathcal{R}$ having a near unanimity polymorphism (NUP).

Our main result

We show that $\mathcal{R}$ having a NUP $\Rightarrow \text{CSP}(\mathcal{R}) \subseteq NL$.

- This is consistent with, but does not prove, the conjecture.
Remarks on the proof

- Intricate and complicated
  - “[It] is a bit of a mess.” (anon. referee)

- Heavily indebted to Dalmau, Krokhin (2008), who proved the result in the case of 3-ary NUPs.

- Like DK, we prove $\text{CSP}(\mathcal{R}) \in \text{NL}$ by establishing a technical property called *bounded pathwidth duality*.

- Even stealing everything we can, our proof introduces significant new complications.

- In particular, we needed to show that a certain algebraic property, which is obvious for 3-ary NUPs, has a surprising (to algebraists) weak analogue in NUPs of higher arities.

Thank you