Tutorial on Universal Algebra, Mal’cev Conditions, and Finite Relational Structures: Lecture II

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Interpretation relation on varieties gives us \( \mathcal{L} \).

Sitting inside \( \mathcal{L} \) is the \( \land \)-closed sub-poset \( \mathcal{ALG}_{\text{fin}} \).

Pp-definability relation on finite structures gives us \( \mathcal{REL}_{\text{fin}} \).

\( \mathcal{REL}_{\text{fin}} \) and \( \mathcal{ALG}_{\text{fin}} \) are anti-isomorphic via \([H] \mapsto [\text{var}(\text{PolAlg}(H))]\).

Mal’cev classes in \( \mathcal{L} \) induce filters on \( \mathcal{ALG}_{\text{fin}} \) and ideals on \( \mathcal{REL}_{\text{fin}} \).
One more set to define:

\[ \mathcal{REL}_{\text{fin}} = \mathcal{ALG}_{\text{fin}} = \{ [H] \in \mathcal{REL}_{\text{fin}} : \text{language of } H \text{ is finite} \} \]

**Convention:** henceforth, all mentioned relational structures under consideration have **finite** languages.
Theorem (Hell, Nešetřil, 1990)

Suppose $G$ is a finite undirected graph (without loops).

- If $G$ is bipartite, then $\text{CSP}(G)$ is in P.
- Otherwise, $\text{CSP}(G)$ is NP-complete.

What the heck is “$\text{CSP}(G)$”?

Definition

Given a finite relational structure $G$ with finite language $L$, the constraint satisfaction problem with fixed template $G$, written $\text{CSP}(G)$, is the following decision problem:

**Input**: an arbitrary finite $L$-structure $I$.

**Question**: does there exist a homomorphism $I \rightarrow G$?

Also called the $G$-homomorphism (or $G$-coloring) problem.
Some context

- [Classical]: CSP($K_2$) $\equiv$ checking bipartiteness, which is in $P$.
  CSP($K_n$) $\equiv$ graph $n$-colorability, which is $NP$-complete for $n \geq 3$ (Karp).

- **Key fact** [Essentially due to Bulatov & Jeavons, unpubl.]:

  If $G, H$ are finite structures in finite languages and $G \prec_{pp} H$, then CSP($G$) is no harder than CSP($H$).

Consequences:

- If CSP($G$) is in $P$ [resp. $NP$-complete], then same is true $\forall H \in [G]$.

- $\{[G] :$ CSP($G$) is in $P\}$ is a down-set in $REL_{\text{fin}}^\omega$.

- $\{[G] :$ CSP($G$) is $NP$-complete$\}$ is an up-set in $REL_{\text{fin}}^\omega$.

- In fact:
  - $\{[G] :$ CSP($G$) is in $P\}$ is an ideal in $(REL_{\text{fin}}^\omega, \lor)$. (Not hard)
Pictorially:

\[ K_3 \]
\[ K_2 \]

\[ \text{non-bipart. graphs} \]
\[ \text{bipart. graphs} \]
\[ \emptyset \]

CSP(-) is NP-complete
CSP(-) is in P

\( \mathcal{REL}_{\text{fin}}^\omega : \)

Hell-Nešetřil theorem: there is \textbf{dichotomy} for undirected graphs.

\textbf{The CSP dichotomy conjecture (Feder, Vardi (1998))}

There is general dichotomy. I.e., for every finite relational structure \( G \) in a finite language, \( \text{CSP}(G) \) is either in \( P \) or is \( NP \)-complete.
Initial steps towards a proof of the Dichotomy Conjecture

1. Reduction to cores.

**Definition**

Let $G, H$ be finite relational structures in the same language.

- $G$ is **core** if all of its endomorphisms are automorphisms.
- $G$ is a **core of** $H$ if $G$ is core and is a retract of $H$.

**Facts:**

- Every finite relational structure $H$ has a core, which is unique up to isomorphism; call it $\text{core}(H)$.
- $\text{CSP}(H) = \text{CSP}(\text{core}(H))$.

Hence when testing dichotomy, we need only consider cores.
2. Reduction to the endo-rigid case.

**Definition**

Let $H = (H, \{relations\})$ be a relational structure.

- $H$ is endo-rigid if its only endomorphism is $\text{id}_H$.
- $H^c := (H, \{relations\} \cup \{\{a\} : a \in H\})$. ("$H$ with constants")

**Facts:**

- Endo-rigid $\Rightarrow$ core.
- $H^c$ is endo-rigid.

**Proposition (Bulatov, Jeavons, Krokhin, 2005)**

If $H$ is core, then $\text{CSP}(H)$ and $\text{CSP}(H^c)$ have the same difficulty.

Hence when testing general dichotomy, we need only consider structures with constants (equivalently, endo-rigid structures).
The reductions in pictures:

\[
\mathcal{REL}_\text{fin}^\omega: \quad [K_3] \rightarrow [G] \rightarrow [H] \rightarrow [H^c]
\]

where \( H = \text{core}(G) \)

\[
\text{CSP}(G), \text{CSP}(H), \text{and CSP}(H^c)
\]

are equally difficult.
“When testing general dichotomy, we need only consider endo-rigid structures.”

\[
\mathcal{REL}_\omega^{\omega} = \bigcup [K_3]
\]

\[
[K_3] = [K_3^\mathcal{E}]
\]

Define \( \mathcal{E} := \bigcup [K_3] = [K_3^\mathcal{E}] \) = \( \{ [H] \in \mathcal{REL}_\omega^{\omega} : H \text{ is endo-rigid} \} \)

\[ \therefore \text{To establish general dichotomy, it suffices to establish dichotomy in } \mathcal{E}. \]

**Question:** Where in \( \mathcal{E} \) should the “dividing line” be?
Consider the situation for graphs.

Hell-Nešetřil explained: for a finite graph $G$,

- $G$ bipartite $\Rightarrow \text{core}(G) = K_2$ or 1.
- $G$ non-bipartite $\Rightarrow \ldots \text{[core}(G)^c] = [K_3]$. 

\[ [K_3] = [\text{core}(G)^c] \]
Question: Where in \( \mathcal{E} \) should the “dividing line” be?

\[
\mathcal{E} = \begin{cases} 
[K_3] & \text{NP-complete} \\
[K_2^\mathcal{E}] & \text{in } P
\end{cases}
\]

The Algebraic CSP Dichotomy Conjecture (BKJ 2000)

We have dichotomy in \( \mathcal{E} \); moreover, the “dividing line” separating \( P \) from \( NP \)-complete is between \( \mathcal{E} \setminus \{[K_3]\} \) and \( \{[K_3]\} \).
Back to algebra: the **Taylor class** $T$.

**Definition**

$T = \text{the class of varieties } V \text{ such that } \exists n \geq 1, \exists \text{ term } t(x_1, \ldots, x_n) \text{ s.t.}$

1. $\forall 1 \leq i \leq n, \exists \text{ an identity of the form}$

   $$V \models t(\text{vars}, x, \ldots, \text{vars}) \approx t(\text{vars}, y, \ldots, \text{vars});$$
   $$\uparrow$$
   $$i \uparrow i$$

2. $V \models t(x, x, \ldots, x) \approx x. \quad \text{ (“} t \text{ is idempotent.”)}$

**Jargon:** such a term $t$ (witnessing $V \in T$) is called a **Taylor term** for $V$.

**Fact:** $T$ forms a filter in $\mathcal{L}$ (and hence is a Mal’cev class).
Theorem (Taylor, 1977)

For any idempotent variety $V$ (i.e., all basic operations are idempotent), either $[V] = [SETS]$ or $V \in T$. 

No idempotent varieties
Now suppose $\mathbf{H}$ is a finite endo-rigid structure.

Then every basic operation of $\text{PolAlg}(\mathbf{H})$ is idempotent.

**Proof:** $f \in \text{Pol}(\mathbf{H}) \Rightarrow f(x, x, \ldots, x)$ is an endomorphism of $\mathbf{H} \\
\Rightarrow f(x, x, \ldots, x) \approx x \quad (\mathbf{H}$ is endo-rigid).

Hence $V := \text{var}(\text{PolAlg}(\mathbf{H}))$ is an idempotent variety.

As $[\mathbf{H}] = [K_3]$ in $\mathcal{E}$ iff $[V] = [\text{SETS}]$ in $\mathcal{L}$, we get

**Corollary**

*Suppose $[\mathbf{H}] \in \mathcal{E}$.  
  
  If $[\mathbf{H}] \neq [K_3]$, then $\text{var}(\text{PolAlg}(\mathbf{H})) \in T$ (i.e., $\mathbf{H}$ has a “Taylor polymorphism”).  
  
  Hence the Algebraic Dichotomy Conjecture is equivalent to 
  
  $\mathbf{H}$ endo-rigid and has a Taylor polymorphism $\Rightarrow \text{CSP}(\mathbf{H}) \in P$.  
  

How close are we to verifying the Algebraic CSP Dichotomy Conjecture?

\[ \mathcal{E} = \begin{array}{c}
\begin{array}{c}
[K_3] \\
\text{known in } P
\end{array}
\end{array} \xrightarrow{[H] \mapsto [V] \text{ where }} V := \text{var}(\text{PolAlg}(H)) \]

\[ = \mathcal{L} \]

- Measure progress (i.e., the portion of \( \mathcal{E} \setminus \{[K_3]\} \) known to be in \( P \)) via its image in \( \mathcal{L} \).
- Thesis: progress is “robust” if its image in \( \mathcal{L} \) “is” a Mal’cev class.
CM = “congruence modular”

HM = “Hobby-McKenzie”

On $\mathcal{A}_{\text{LG}_{\text{fin}}}$: omit types 1,5

SD($\land$) = “congruence meet-semidistributive”

On $\mathcal{A}_{\text{LG}_{\text{fin}}}$: omit types 1,2

T = “Taylor”

On $\mathcal{A}_{\text{LG}_{\text{fin}}}$: omit type 1
Another theme: finding “good” Taylor terms.

**Definition**

An operation $f$ of arity $k \geq 2$ is called a **WNU** operation if it satisfies

\[ f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx f(x, x, y, \ldots, x) \approx \cdots \]

and

\[ f(x, x, \ldots, x) \approx x. \]

**Observe:** any WNU is a Taylor operation.

**Theorem (Maróti, McKenzie, 2008, verifying a conjecture of Valeriote)**

Suppose $A$ is a finite algebra and $V = \text{var}(A)$. If $V$ has a Taylor term, then $V$ has a WNU term.
Definition

An operation $f$ of arity $k \geq 2$ is called a **cyclic** operation if it satisfies

$$f(x_1, x_2, x_3, \ldots, x_k) \approx f(x_2, x_3, \ldots, x_k, x_1)$$

and

$$f(x, x, \ldots, x) \approx x.$$

Observe: any cyclic operation is a WNU, since we can specialize the first identity to get

$$f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx f(x, x, y, \ldots, x) \approx \cdots.$$  

Theorem (Barto, Kozik, 201?)

Suppose $A$ is a finite algebra and $V = \text{var}(A)$. If $V$ has a Taylor term, then $V$ has a cyclic term. (In fact, has a $p$-ary cyclic term for every prime $p > |A|$.)
Easy proof of the Hell-Nešetřil theorem, using cyclic terms.
Due to Barto, Kozik?

Let $G = (G, E)$ be a finite graph; assume that it is core and not bipartite.

We must show that $[G^c] = [K_3]$.

Assume the contrary. Then $G^c$ (and hence also $G$) has a Taylor polymorphism.

So by the Barto-Kozik theorem, $G$ has a cyclic polymorphism of arity $p$ for every prime $p > |G|$.

$G$ not bipartite $\Rightarrow G$ contains an odd cycle, and hence contains cycles of every odd length $> |G|$.
Pick a prime $p > |G|$ and a cycle $a_1, a_2, \ldots, a_p$ in $G$ of length $p$. That is,

$$(a_1, a_2), (a_2, a_3), \ldots, (a_{p-1}, a_p), (a_p, a_1) \in E.$$ 

Pick a cyclic polymorphism $f$ of $G$ of arity $p$.

Observe that if

$$u = (a_1, a_2, \ldots, a_{p-1}, a_p)$$
$$v = (a_2, a_3, \ldots, a_p, a_1),$$

then $(u, v)$ is an edge of $G^p$.

As $f$ is a homomorphism $G^p \rightarrow G$, we get that $(f(u), f(v))$ is an edge of $G$.

But $f(u) = f(v)$ because $f$ is cyclic. So $(f(u), f(v))$ is a loop.

Contradiction!!
In conclusion:

- Good progress is being made on the CSP Dichotomy Conjecture, with essential help from universal algebra.
- The conjecture is motivating new purely algebraic conjectures, some of which have been recently proved.

Thank you!