AN OVERVIEW OF MODERN UNIVERSAL ALGEBRA

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Abstract. This article, aimed specifically at young mathematical logicians, gives a gentle introduction to some of the central achievements and problems of universal algebra during the last 25 years.

I intend in this article to introduce nonspecialists to the fact that there are deep results in contemporary universal algebra. The first four sections give the context in which universal algebra has something to say, and describe some of the basic results upon which much of the work in the field is built. Section 5 covers the highlights of tame congruence theory, a sophisticated point of view from which to analyze locally finite algebras. Section 6 describes some of the field's "big" results and open problems concerning finite algebras, notably the undecidability of certain finite axiomatizability problems and related problems, and the so-called "RS problem," currently the most important open problem in the field.

This article presents a personal view of current universal algebra, one which is limited both by my ignorance of large parts of the field as well as the need to keep the article focused. For example, I do not mention natural duality theory, one of the most vigorous subdisciplines of the field, nor algebraic logic or work that is motivated by and serves computer science. Therefore the views expressed in this article should not be taken to be a comprehensive statement of "what universal algebra is."

The text is frequently imprecise and proofs, when present, are merely sketched. References are often omitted. My aim is to give the reader an impression of the field. Resources for further reading are provided in the final section.

1. Algebras and equational classes

The fundamental objects of study in universal algebra are *algebraic structures*, or *algebras* for short. These are structures in the traditional sense of model theory but with no relations. The algebras in this article can be assumed with no loss of generality to have a countable language, and often the language will be finite. Notationally I will use $\mathbf{A}$ to denote an algebra with universe A, and will write $\mathbf{A} = \langle A; f_0, f_1, \ldots\rangle$. Unlike model theory, which is primarily interested in the structure

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$\langle A; \{\text{all definable sets of } \mathbf{A}\}\rangle$, universal algebra might be described as the study of the more mundane $\langle A; \{\text{all term functions of } \mathbf{A}\}\rangle$.

Universal algebra from the very beginning has focused on the *equations* $\forall \overline{x} : t_1(\overline{x}) = t_2(\overline{x})$ (also called *identities*) modeled by an algebra. This focus immediately gives rise in the usual fashion to the galois connection between *equational theories* (set of all equations modeled by an algebra, or by a class of algebras) and *equational classes* (class of all models of an equational theory). The latter are usually called *varieties* by universal algebraists but I will avoid this usage in this article.

While the basic constructions for model theory might be ultraproducts and elementary substructures, the basic constructions for universal algebra are simpler ones which preserve equations, including:

(1) *Homomorphic images* (i.e., images of surjective maps which preserve atomic formulas).
(2) *Subalgebras* (i.e., substructures).
(3) *Products* (i.e., full cartesian products over arbitrary index sets).

$\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ denote closure (of a class of algebras in the same language) under homomorphic images, subalgebras and products respectively. Two basic theorems concerning these closure operators are:

Theorem 1.1. *(G. Birkhoff) A class $\mathcal{K}$ of algebras in the same language is an equational class if and only if it is closed under* $\mathbf{H}$*,* $\mathbf{S}$ *and* $\mathbf{P}$*.*

Theorem 1.2. *(A. Tarski) Suppose $\mathcal{K}$ is a class of algebras in the same language. Then the smallest equational class containing $\mathcal{K}$ is* $\mathbf{HSP}(\mathcal{K})$*.*

There has been long-standing fascination with equational classes $\mathbf{HSP}(\mathbf{A})$ generated by a finite algebra $\mathbf{A}$. Perhaps the fascination stemmed from particular equational classes that were seen to be significant from the early days of universal algebra, such as:

(1) The class of boolean algebras. This class is known to be the equational class generated by the 2-element boolean algebra:
$$\{\text{boolean algebras}\} = \mathbf{HSP}(\langle 2; \wedge, \vee, ', 0, 1\rangle).$$
(2) The class of distributive lattices. This class is known to be the equational class generated by the 2-element (distributive) lattice:
$$\{\text{distributive lattices}\} = \mathbf{HSP}(\langle 2; \wedge, \vee\rangle).$$

If an equational class has a finite language, one can ask if the class is finitely axiomatizable. Universal algebraists have been particularly interested in asking this question when the class is generated by a finite algebra. Two enduring problems are:

Finite Basis Problem (1950's). For which finite algebras $\mathbf{A}$ with a finite language is $\mathbf{HSP}(\mathbf{A})$ finitely axiomatizable?

Tarski's Finite Basis Problem (1960's). Is the answer to the Finite Basis Problem recursive?

The first problem has motivated a great deal of work over the last 50 years and is considered to be intractable in general but possibly solvable when restricted to well-behaved classes of algebras. I will return to these problems later in this article.

More recent work, meaning since 1980 or so, has largely focused on structural or "representational" questions. In general terms, the emphasis is on the study and classification of equational classes, particularly those generated by a finite algebra, according to the nature of the algebraic and combinatorial structure of their members.

2. Congruences

Let h be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, i.e., a map $h : A \to B$ which preserves atomic formulas. h partitions the universe of $\mathbf{A}$ into blocks $h^{-1}(b)$, $b \in h(A)$. Such a partition induced by a homomorphism is called a *congruence* of $\mathbf{A}$. Natural examples of congruences include the partition of a group given by the cosets of a normal subgroup, or the partition of a ring given by the cosets of an ideal. As in the case of groups and rings, there is an intrinsic definition of congruence which does not make reference to homomorphisms.

Given a congruence α of an algebra $\mathbf{A}$, there is a natural way to define a *quotient algebra* $\mathbf{A}/\alpha$; the universe of this algebra is the set of blocks of α, and $\mathbf{A}/\alpha$ itself is a homomorphic image of $\mathbf{A}$. The standard homomorphism and isomorphism theorems of group and ring theory have their counterparts in this general setting.

Fix an algebra $\mathbf{A}$. We shall consider the set $\operatorname{Con}\mathbf{A}$ of all congruences of $\mathbf{A}$. For congruences α, β of $\mathbf{A}$, we write $\alpha \leq \beta$ if α is a refinement of β, i.e., each α-block is contained in a single β-block. It turns out that $\langle \operatorname{Con}\mathbf{A}; \leq \rangle$ is a *bounded lattice*, i.e., with respect to the ordering $\leq$, $\operatorname{Con}\mathbf{A}$ has a greatest element 1 and a least element 0, and any two congruences α, β of $\mathbf{A}$ have a least upper bound $\alpha \vee \beta \in \operatorname{Con}\mathbf{A}$ and a greatest lower bound $\alpha \wedge \beta \in \operatorname{Con}\mathbf{A}$. The greatest element 1 is the partition of A having only the one block A, while the least element 0 is the partition of A in which every block has only one element.

The congruence lattice of an algebra is typically the first point of reference when a universal algebraist seeks to understand the structure of the algebra. Just as a ring theorist might be interested in the maximal ideals of a ring, a universal algebraist might need to understand the maximal congruences of an algebra. Where a group theorist might consider a composition series of a finite group, a universal algebraist will consider a maximal increasing chain in the congruence lattice of a finite algebra.

Something that seems unique to universal algebra, however, is the observation that the possession by an algebra of term functions satisfying certain equations is connected to restrictions on the *shape* of the congruence lattice of the algebra. To illustrate, I offer two samples. Recall that the *distributive law* for lattices is

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

while the *modular law* is the restriction of this law to those instances with $y \leq x$.

Theorem 2.1. *Suppose* $\mathbf{A}$ *has a ternary term function* $p(x, y, z)$ *such that* $\mathbf{A} \models \forall x, y : \; p(x, x, y) = p(y, x, x) = y$. *Then* $\operatorname{Con} \mathbf{A}$ *satisfies the modular law.*

As an example, the term $(x \cdot y^{-1}) \cdot z$ defines a Mal'cev term function in any group, and hence the congruence lattice (= normal subgroup lattice) of any group is modular.

Theorem 2.2. *Suppose* $\mathbf{A}$ *has a* majority term function, *i.e., a ternary term function* $m(x, y, z)$ *such that* $\mathbf{A} \models \forall x, y : \; m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. *Then* $\operatorname{Con} \mathbf{A}$ *satisfies the distributive law.*

As an example, the term $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ defines a majority term function in any boolean algebra, and hence the congruence lattice (= lattice of filters) of any boolean algebra is distributive.

3. Irreducible algebras

A *representation* of an algebra $\mathbf{A}$ is a collection $\{h_i : i \in I\}$ of homomorphisms with domain $\mathbf{A}$ which collectively separate the points of A. That is, if $a, b \in A$ with $a \neq b$ then there must exist $i \in I$ such that $h_i(a) \neq h_i(b)$.

The point of a representation is that it provides a way to embed the algebra into a product of potentially smaller algebras. For suppose $\{h_i : i \in I\}$ is a representation of $\mathbf{A}$ with $h_i : \mathbf{A} \to \mathbf{B}_i$. Then the map $a \mapsto (h_i(a))_{i \in I}$ embeds $\mathbf{A}$ into $\prod_{i \in I} \mathbf{B}_i$. If each h_i is surjective (which can be assumed), then each $\mathbf{B}_i$ is a homomorphic image of $\mathbf{A}$. Thus a representation of an algebra gives a way to realize it as a subalgebra of a product of (some of) its homomorphic images. One would like these images to be simpler than $\mathbf{A}$ itself. This will certainly not be the case if one of the images is $\mathbf{A}$ itself, i.e., if some h_i is injective. Thus we say that a representation $\{h_i : i \in I\}$ of $\mathbf{A}$ is *trivial* if some h_i is injective.

An algebra is *irreducible* (or *subdirectly irreducible*) if every representation of it is trivial. For example, the ring $\mathbb{Z}_n$ is irreducible if and only if n is a prime power. More generally, a ring is irreducible if and only if it has a least nonzero ideal, i.e., a nonzero ideal which is contained in every nonzero ideal. This generalizes beyond rings:

Theorem 3.1. *An algebra having more than one element is irreducible if and only if it has a least nonzero congruence.*

Equational classes are generated by their irreducible members, as a consequence of Theorem 1.2 and the following theorem.

Theorem 3.2. *(G. Birkhoff) For every algebra* $\mathbf{A}$ *having more than one element there exists a representation* $\{h_i : i \in I\}$ *of* $\mathbf{A}$ *such that each* h_i *is a surjective homomorphism from* $\mathbf{A}$ *onto an irreducible homomorphic image of* $\mathbf{A}$*.*

To close this section, we connect the concepts introduced so far by stating and proving the following theorem, whose proof (except for the first paragraph) was given by A. Foster and A. Pixley in the 1960's.

Theorem 3.3. *Let* $\mathbf{A}$ *be a finite algebra. If* $\operatorname{Con}\mathbf{B}$ *is distributive for every* $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$*, then every irreducible member of* $\mathbf{HSP}(\mathbf{A})$ *has cardinality at most* $|A|$*.*

Proof sketch. It can be shown that every irreducible algebra $\mathbf{D}$ can be embedded in an ultraproduct of finitely generated irreducible members of $\mathbf{HS}(\mathbf{D})$. Hence it suffices to restrict our attention to the finitely generated irreducible members of $\mathbf{HSP}(\mathbf{A})$.

Let $n = |A|$ and let $\mathbf{D}$ be a finitely generated irreducible member of $\mathbf{HSP}(\mathbf{A})$. Because $\mathbf{A}$ is finite, $\mathbf{HSP}(\mathbf{A})$ is *locally finite* (i.e., every finitely generated subalgebra of a member of $\mathbf{HSP}(\mathbf{A})$ is finite), so in particular $\mathbf{D}$ is finite.

We have $\mathbf{D} \in \mathbf{HSP}(\mathbf{A})$, which means there exist a nonempty set I, a subalgebra $\mathbf{B}$ of $\mathbf{A}^I$, and a congruence δ of $\mathbf{B}$ with $\mathbf{D} \cong \mathbf{B}/\delta$. We may assume $\mathbf{D} = \mathbf{B}/\delta$. Because $\mathbf{D}$ is finite, we may also assume that $\mathbf{B}$ is finitely generated and hence is finite.

Because $\mathbf{D}$ is irreducible, it has a least nonzero congruence μ. Because $\mathbf{D} = \mathbf{B}/\delta$, the congruences of $\mathbf{D}$ correspond exactly to the congruences of $\mathbf{B}$ above δ. Thus $\mathbf{B}$ has a least congruence ν strictly above δ. Since $\operatorname{Con}\mathbf{B}$ is a distributive lattice by assumption, it follows that δ is *meet-prime* in $\operatorname{Con}\mathbf{B}$; that is, if $\rho_0, \dots, \rho_{N-1}$ are congruences of $\mathbf{B}$ and $\rho_0 \wedge \cdots \rho_{N-1} \leq \delta$, then there exists $i < N$ such that $\rho_i \leq \delta$.

(Proof: assuming $\rho_0 \wedge \cdots \rho_{N-1} \leq \delta$ we have

$$\begin{aligned} \delta &= \delta \vee (\rho_0 \wedge \cdots \wedge \rho_{N-1}) \\ &= (\delta \vee \rho_0) \wedge \cdots \wedge (\delta \vee \rho_{N-1}) \end{aligned}$$

where the last line is an application of the distributive law in $\operatorname{Con}\mathbf{B}$. Recall that ν is the least congruence of $\mathbf{B}$ strictly above δ. Thus for each $i < N$, since $\delta \leq \delta \vee \rho_i$, either $\delta = \delta \vee \rho_i$ or $\nu \leq \delta \vee \rho_i$. If $\delta \neq \delta \vee \rho_i$ for every $i < N$, then ν would be a common lower bound to all the $\delta \vee \rho_i$ and hence

$$\delta < \nu \leq (\delta \vee \rho_0) \wedge \cdots \wedge (\delta \vee \rho_{N-1})$$

which would contradict the previous displayed equations. Therefore there exists $i < N$ such that $\delta = \delta \vee \rho_i$, which is equivalent to $\rho_i \leq \delta$.)

Because $\mathbf{B}$ is a subalgebra of $\mathbf{A}^I$, there is a representation $\{h_i\}$ of $\mathbf{B}$ consisting of homomorphisms from $\mathbf{B}$ into $\mathbf{A}$. As $\mathbf{B}$ is finite, the set $\{h_i\}$ can be taken to be finite, say $\{h_i\} = \{h_i : i < N\}$. For each $i < N$ let ρ_i be the congruence of $\mathbf{B}$ induced by h_i. As $\{h_i : i < N\}$ is a representation of $\mathbf{B}$, we have $\rho_0 \wedge \cdots \wedge \rho_{N-1} = 0$ in $\operatorname{Con}\mathbf{B}$. Hence $\rho_0 \wedge \cdots \rho_{N-1} \leq \delta$. As δ is meet-prime, there exists $i < N$ such that $\rho_i \leq \delta$. Hence ρ_i is a refinement of δ, so δ has no more blocks than ρ_i. As the number of blocks of δ is equal to $|D|$ (because $\mathbf{D} = \mathbf{B}/\delta$) while the number of blocks of ρ_i is equal to $|h_i(B)|$ and hence is at most n (because $h_i : \mathbf{B} \to \mathbf{A}$ and $|A| = n$), it follows that $|D| \leq n$. □

4. Abelian congruences and congruence modular equational classes

An algebra $\mathbf{A}$ is said to be *abelian* if it models the following infinite family of universal Horn sentences: for every term $f(\overline{x}, \overline{y})$ in the language of $\mathbf{A}$,

$$\forall \overline{a}, \overline{b}, \overline{c}, \overline{d} : \; f(\overline{a}, \overline{c}) = f(\overline{a}, \overline{d}) \leftrightarrow f(\overline{b}, \overline{c}) = f(\overline{b}, \overline{d}).$$

For example:

(1) Sets are abelian.
(2) R-modules (construed as abelian groups with endomorphisms named by the elements of R) are abelian.
(3) A group is abelian in the above sense if and only if it is abelian in the usual sense.
(4) No ring with identity having more than one element is abelian.
(5) In equational classes in which every member has a distributive congruence lattice, no algebra having more than one element is abelian.

This definition of "abelian" is not entirely well-behaved. For example, it is not generally true that homomorphic images of abelian algebras are abelian; in fact, *every* algebra is a homomorphic image of some abelian algebra. Nonetheless, it turns out to be a surprisingly useful notion in some contexts, as we shall see.

One can relativize the notion of abelianness to congruences. One does this by restricting the quantified parameters $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ to those instances in which each pair a_i, b_i belong to a common block of the congruence, and likewise each pair c_j, d_j belong to a common block of the congruence. Then:

(1) Every congruence of an abelian algebra is abelian.
(2) A congruence of a group is abelian if and only if the normal subgroup corresponding to it is abelian in the usual sense.
(3) A congruence of a ring is abelian if and only if the ideal corresponding to it annihilates itself.
(4) In congruence distributive equational classes, only the zero congruence is abelian.

I have already defined the modular law for lattices and observed that it is a weaker than the distributive law. An equational class is *congruence modular* if $\operatorname{Con} \mathbf{A}$ satisfies the modular law for every algebra $\mathbf{A}$ in the class. It turns out that essentially all equational classes of "classical" algebras (e.g., groups, rings, modules, boolean algebras, lattices) are congruence modular. Thus it was with some excitement in the 1980's that the community uncovered a rich and deep theory of the good behavior of the notion of abelianness in congruence modular equational classes [35, 10, 9, 8]. (For example, abelian algebras in this context are definitionally equivalent to R-modules, and homomorphic images of abelian algebras are abelian.) This theory is called *modular commutator theory*. At its most basic level it reveals how, in congruence modular equational classes, the abelian/nonabelian dichotomy mediates tight connections between (1) the term functions of $\mathbf{A}$, and (2) the local shape of $\operatorname{Con}\mathbf{A}$. Somewhat more precisely,

(1) Abelian congruences restrict the complexity of term functions: the blocks support translations of linear functions over modules only.
(2) Nonabelian congruences enforce consequences of distributivity near the congruence.

Here is an example of what I mean by "enforcing consequences of distributivity" in the nonabelian case. Suppose $\mathbf{D}$ is an irreducible algebra with least nonzero congruence μ. Suppose moreover that $\mathbf{D} = \mathbf{B}/\delta$ for some algebra $\mathbf{B}$ and some congruence $\delta \in \operatorname{Con}\mathbf{B}$. Then in this situation, if $\mathbf{B}, \mathbf{D}$ belong to a congruence modular equational class and μ is nonabelian in $\operatorname{Con}\mathbf{D}$, then δ is meet-prime in $\operatorname{Con}\mathbf{B}$. Since this fact was the only consequence of distributivity used in the proof of Theorem 3.3, we have the following corollary.

Corollary 4.1. *Let* $\mathbf{A}$ *be a finite algebra. If* $\mathbf{HSP}(\mathbf{A})$ *is congruence modular, then every irreducible member of* $\mathbf{HSP}(\mathbf{A})$ *whose least nonzero congruence is nonabelian has cardinality at most* $|A|$.

The previous result is the first step in proving the following deep result about the irreducible members of finitely generated congruence modular varieties.

Theorem 4.2. *(R. Freese, R. McKenzie* [7]*) If* $\mathbf{A}$ *is a finite algebra and* $\mathbf{HSP}(\mathbf{A})$ *is congruence modular, then*

(1) *(Gap) Either there is a finite bound to the cardinalities of the irreducibles in* $\mathbf{HSP}(\mathbf{A})$*, or there is no bound at all.*
(2) *(Recursive) If the language of* $\mathbf{A}$ *is finite, then one can effectively determine from* $\mathbf{A}$ *which alternative holds.*

Proof sketch. By the previous result, it suffices to consider only the irreducibles in $\mathbf{HSP}(\mathbf{A})$ with abelian least nonzero congruence. If such an irreducible has a nonabelian congruence "independent from" the least nonzero congruence (in a certain technical sense), then a uniform construction produces irreducibles in $\mathbf{HSP}(\mathbf{A})$ of every infinite cardinality. If no irreducible has such an independent nonabelian congruence, then the structure of each irreducible is relatively well understood. In particular, on the blocks of a large abelian congruence of the irreducible one can build a many-sorted irreducible module (each block an abelian group, each operation linear) in which (i) the number of sorts and (ii) the number of operations are finite and uniformly bounded. This implies a finite uniform bound to the size of these irreducibles, which moreover is computable from $\mathbf{A}$ if the language of $\mathbf{A}$ is finite. $\square$

This theorem and other data convinced D. Hobby and McKenzie to formulate in the 1980's what we call here the RS (gap) conjecture; its recursive companion was implicitly accepted as an inevitable consequence of the eventual proof of the conjecture.

RS Conjecture 4.3. For any finite algebra $\mathbf{A}$,

(1) (Gap) Either there is a finite bound to the cardinalities of the irreducibles in $\mathbf{HSP}(\mathbf{A})$, or there is no bound at all.
(2) (Recursive) If the language of $\mathbf{A}$ is finite, then one can effectively determine from $\mathbf{A}$ which alternative holds.

5. Elements of tame congruence theory

Also in the 1980's, Hobby and McKenzie developed a remarkable, exotic, deep tool for understanding fine structure in finite algebras. The analysis is essentially combinatorial and is built on the idea of "localization at a minimal nonzero congruence." This section introduces the reader to the basic definitions and ideas of the theory.

5.1. **Polynomial subreducts.** Let $\mathbf{A}$ be an algebra. A *polynomial function* of $\mathbf{A}$ is a term function allowing parameters from A. For any nonempty subset $S \subseteq A$ we define a new algebra $\mathbf{A}|_S$ (in a new language) as follows:

- The universe of $\mathbf{A}|_S$ is S.
- The functions of $\mathbf{A}|_S$ are all restrictions to S of n-ary polynomial functions f of $\mathbf{A}$ (all n) which satisfy $f(S^n) \subseteq S$.

This new algebra $\mathbf{A}|_S$ is called the *polynomial algebra induced on S* (by $\mathbf{A}$).

Example 5.1. Let $\mathbf{V}$ be a 1-dimensional vector space over $GF(4)$. The polynomial functions of $\mathbf{V}$ are all n-ary functions of the form

$$\sum_{i=1}^{n} \lambda_i x_i + a, \ \ \lambda_i \in GF(4), \ \ a \in V.$$

Let $S = V \setminus \{0\}$. Then every function of $\mathbf{V}|_S$ is either constant or depends on only one variable, and in the latter case is either the identity map or a cyclic permutation of S applied to that variable. Let $\mathbf{S}_1$ be the algebra $\langle S;\ g(x)\ (g \in C_3)\rangle$ where C_3 is the cyclic permutation group of order 3 acting on S. $\mathbf{S}$ is an example of a G-set (with $G = C_3$). Then $\mathbf{V}|_S$ is *polynomially equivalent* to $\mathbf{S}$, meaning that $\mathbf{V}|_S$ and $\mathbf{S}$ have the same universe and the same polynomial functions.

Example 5.2. Start with the alternating group $\mathbf{A}_4$ and let V be its unique 4-element normal subgroup. We shall describe the polynomial algebra induced on V by $\mathbf{A}_4$. Let $\mathbf{V}$ be the abelian group V expanded to become a 1-dimensional vector space over $GF(4)$. The polynomial functions of $\mathbf{V}$ were described in the previous example. It can be shown that $\mathbf{A}_4|_V$ is polynomially equivalent to $\mathbf{V}$.

5.2. **Idempotent ranges, minimal sets and traces.** Tame congruence theory exploits information contained in polynomial algebras induced on certain sets associated with congruences or pairs of congruences, and especially certain sets called *minimal sets*, *bodies* and *traces* associated with a minimal nonzero congruence.

Definition 5.3. Fix a finite algebra $\mathbf{A}$ and a minimal nonzero congruence α of $\mathbf{A}$.

(1) An *idempotent* of $\mathbf{A}$ is a unary polynomial function $e(x)$ of $\mathbf{A}$ such that $\mathbf{A} \models \forall x: \ e(e(x)) = e(x)$.
(2) An *idempotent range* of $\mathbf{A}$ is the range of any idempotent of $\mathbf{A}$.
(3) $\mathcal{N}_\alpha$ is the set of those idempotent ranges of $\mathbf{A}$ whose intersection with at least one α-block has more than one element.
(4) An α-*minimal set* is any member of $\mathcal{N}_\alpha$ which is minimal among all members of $\mathcal{N}_\alpha$ with respect to inclusion.
(5) An α-*trace* is any intersection of an α-minimal set with an α-block, provided the intersection has at least two elements.
(6) An α-*body* is the union of all the α-traces contained in some α-minimal set.

Example 5.4. Consider the alternating group $\mathbf{A}_4$. It has exactly three normal subgroups: A_4, $\{1\}$, and V. Hence it has exactly three congruences: 1, 0 and the congruence α whose blocks are the cosets of V. $\mathbf{A}_4$ has several hundred idempotents and idempotent ranges. Let us explore a tiny fraction of them. Fix an element $a \in A_4$ of order 3. Define

$$\begin{aligned} e_1(x) &= x^4 & \qquad U_1 &= e_1(A_4) \\ e_2(x) &= a(a(ax^4)^4)^4 & \qquad U_2 &= e_2(A_4) \\ e_3(x) &= a(a^{-1}x)^3 & \qquad U_3 &= e_3(A_4). \end{aligned}$$

Since each element of V has order 2 or 1 while each element of $A_4 \setminus V = aV \cup a^2V$ has order 3, it is easy to see that $e_1(x) = x$ for all $x \in \{1\} \cup aV \cup a^2V$ while $e_1(x) = 1$ for $x \in V \setminus \{1\}$. In particular, e_1 is idempotent, $U_1 = \{1\} \cup aV \cup a^2V$, and U_1 is an idempotent range.

Using the above description of e_1 it can be deduced that e_2 maps V to 1, aV to a and a^2V to a^2. Hence e_2 is an idempotent, $U_2 = \{1, a, a^2\}$, and U_2 is an idempotent range. (In fact, every transversal of the cosets of V is an idempotent range.)

Again by considering orders of elements, it is easy to see that $e_3(x) = x$ for $x \in aV$ while $e_3(x) = a$ for $x \in V \cup a^2V$. Hence e_3 is idempotent, $U_3 = aV$, and U_3 is an idempotent range. (Similarly, V and a^2V are idempotent ranges.)

$U_2 \notin \mathcal{N}_\alpha$ since U_2 does not meet any α-block in more than one element. By contrast, both U_1 and U_3 are in $\mathcal{N}_\alpha$. U_1 is not an α-minimal set because it properly contains another member of $\mathcal{N}_\alpha$, for example U_3. Finally, it can be shown (better: a computer can show) that U_2 is an α-minimal set. In fact, the cosets of V are precisely the α-minimal sets of $\mathbf{A}_4$.

Since each α-minimal set in this example is contained entirely inside an α-block, the α-traces and α-bodies are identical to the α-minimal sets, i.e., the cosets of V.

The example just considered is rather simpler than what one should expect. In general, α-traces are *not* entire α-blocks, but subsets of α-blocks. α-traces are *not* identical to α-minimal sets, because α-minimal sets will *not* be contained entirely inside a single α-block. An α-minimal set may be strictly larger than its α-body, and an α-body may consist of more than one α-trace.

5.3. **The five-fold classification.** The notions of α-minimal set and α-trace may seem at first sight to be rather ad hoc. The first hint of their importance appears in the following theorem of P.P. Pálfy [34].

Theorem 5.5. *Let* $\mathbf{A}$ *be a finite algebra,* α *a minimal nonzero congruence of* $\mathbf{A}$*, and* N *an* α*-trace. Then the induced polynomial algebra* $\mathbf{A}|_N$ *is polynomially equivalent to one of:*

(1) *A primitive permutation group construed as a* G*-set (as in Example 5.1), or a 2-element set with no functions.*
(2) *A 1-dimensional vector space over a finite field.*
(3) *A 2-element boolean algebra.*
(4) *A 2-element lattice* $\langle 2; \wedge, \vee \rangle$*.*
(5) *A 2-element semilattice* $\langle 2; \wedge \rangle$*.*

The second hint of the importance of α-traces is the following theorem of Hobby and McKenzie.

Theorem 5.6. [13, Cor. 5.2(2)] *Let* $\mathbf{A}$ *be a finite algebra and* α *a minimal nonzero congruence of* $\mathbf{A}$*. The polynomial algebras induced on any two* α*-traces are isomorphic.*

The last sentence needs amplification, since it is problematic to compare the languages of different induced polynomial algebras. The proper interpretation is that *there exists an identification of their languages with respect to which* the induced polynomial algebras are isomorphic.

Taken together, the two previous theorems assign to each minimal nonzero congruence α of a finite algebra an invariant, namely, the unique (up to isomorphism) primitive permutation group as a G-set, 1-dimensional vector space, or 2-element boolean algebra, lattice, semilattice or set with no functions which is isomorphic to the polynomial algebra induced on any α-trace. We declare the *type* of α to be the numerical label (1–5) of the item in Theorem 5.5 which contains this invariant. For example, the unique minimal congruence of the alternating group $\mathbf{A}_4$ considered in Example 5.4 is "Type 2." In general, for every finite algebra $\mathbf{A}$, the minimal nonzero congruences of $\mathbf{A}$ are each assigned a type from $\{1, 2, 3, 4, 5\}$. This is analogous to the abelian/nonabelian dichotomy for congruences described in Section 4, except that (i) the classification has five flavors rather than two; (ii) it applies only to finite algebras; and (iii) it applies only to minimal nonzero congruences of those algebras.

Theorem 5.6 remains true if "α-trace" is replaced with "α-body" or "α-minimal set." For reasons that will not be fully explained in this article, understanding the structure of the polynomial algebras induced on α-bodies and α-minimal sets is fundamental to applications the theory. However, these polynomial algebras are *not* classified up to isomorphism in all cases; in particular, there is no analogue of Theorem 5.5 for α-minimal sets.

5.4. **Globalization (Part 1).** The type of a minimal nonzero congruence of a finite algebra describes the *local* structure of the algebra within blocks of the congruence. To be useful, this local structure must reflect the global structure of the algebra. One mechanism that ties the local and global structures together is the "geometry" of α-traces and α-minimal sets with respect to polynomial functions. The following result is a rudimentary statement of the situation.

Theorem 5.7. [13, Th. 2.8, Ex. 2.9(2)] *Let* $\mathbf{A}$ *be a finite algebra and* α *a minimal nonzero congruence of* $\mathbf{A}$.

(1) Connectedness. *Each α-block having more than one element is the connected union of α-traces.*
(2) Isomorphism. *For any two α-traces N, N' there exist unary polynomial functions f, g of $\mathbf{A}$ such that $f|_N$ is a bijection from N to N' and $g|_{N'} = (f|_N)^{-1}$. Moreover, there is an identification of the languages of $\mathbf{A}|_N$ and $\mathbf{A}|_{N'}$ with respect to which $f|_N$ and $g|_{N'}$ are mutually inverse isomorphisms.*
(3) Density. *Any two distinct elements a, b of an α-block can be mapped by some unary polynomial function f of $\mathbf{A}$ to distinct elements of an α-trace. Moreover, f can be taken to be an idempotent of $\mathbf{A}$ whose range is an α-minimal set.*

Using slightly more information than this, one can prove the following.

Theorem 5.8. [13, Th. 5.5, 5.6] *Let* $\mathbf{A}$ *be a finite algebra and* α *a minimal nonzero congruence. Then* α *is abelian if and only if its type is 1 or 2.*

Proof sketch. If the type of α is 3, 4 or 5, and N is an α-trace, then $\mathbf{A}|_N$ is a 2-element boolean algebra, lattice or semilattice and hence is nonabelian. Since the functions of $\mathbf{A}|_N$ are derived from those of $\mathbf{A}$, the failure of $\mathbf{A}|_N$ to be abelian also witnesses the failure of α to be an abelian congruence of $\mathbf{A}$.

Conversely, assume the type of α is 1 or 2. The geometry of α-traces would in this case permit a failure of the abelian condition for α to be localized in an α-trace N and hence in the induced algebra $\mathbf{A}|_N$. But $\mathbf{A}|_N$ is either a vector space, a G-set, or a set with no functions, all of which are abelian. Hence α must be abelian. $\square$

5.5. **Generalization to non-minimal congruences.** There is a simple way to extend the above classification to non-minimal congruences. Suppose $\mathbf{A}$ is a finite algebra and let δ, α be congruences of $\mathbf{A}$ with $\delta < \alpha$ and such that α is minimal among all congruences strictly greater than δ. Such a pair (δ, α) is called a *cover pair* (or *prime quotient*). Given the cover pair (δ, α), the larger congruence α corresponds in the standard way to a minimal nonzero congruence $\overline{\alpha}$ of $\mathbf{A}/\delta$. We can then assign to the cover pair (δ, α) the type 1–5 of $\overline{\alpha}$ in $\mathbf{A}/\delta$. In this way every cover pair in $\operatorname{Con}\mathbf{A}$ is assigned one of five types.

Many applications of the theory require information about cover pairs (δ, α) that is not available in the quotient algebra $\mathbf{A}/\delta$. Additional information of interest to us is captured in the polynomial algebras induced on certain subsets of A which, when collapsed by δ, become minimal sets, bodies or traces for $\overline{\alpha}$ in $\mathbf{A}/\delta$. We define a (δ, α)-*minimal set* to be a subset U of A which (i) is the range of an idempotent of $\mathbf{A}$, (ii) becomes an $\overline{\alpha}$-minimal set of $\mathbf{A}/\delta$ when collapsed by δ, and (iii) is minimal under inclusion with respect to properties (i) and (ii). A (δ, α)-*trace* is an intersection of a (δ, α)-minimal set with an α-block, provided the intersection meets at least two distinct δ-blocks. A (δ, α)-body is the union of all (δ, α)-traces contained in a (δ, α)-minimal set.

The polynomial algebras induced on (δ, α)-traces by $\mathbf{A}$ are completely understood in the type 3 and 4 cases only; in the remaining three cases, they are largely understood, and of course are completely understood modulo δ (as their quotients modulo δ are the polynomial algebras induced on $\overline{\alpha}$-traces by $\mathbf{A}/\delta$ and these are described in Theorem 5.5). Less is known of the polynomial algebras induced on (δ, α)-bodies and (δ, α)-minimal sets. When applications demand it, one must occasionally make a detailed study of these induced algebras.

5.6. **Globalization (Part 2).** In addition to the geometry of traces and minimal sets, there are two more elements of the theory that link the local and global structures of a finite algebra. These elements relate two of the most basic notions of universal algebra – congruences and equations – between $\mathbf{A}$ and polynomial algebras induced on minimal sets, bodies and traces by $\mathbf{A}$.

Lemma 5.9. [13, Lem. 2.3] *Let $\mathbf{A}$ be an algebra, U the range of an idempotent of $\mathbf{A}$, and S any nonempty subset of A.*

1. *The map $\theta \mapsto \theta|_S$ is a $\wedge$-preserving map from* $\operatorname{Con}\mathbf{A}$ *into* $\operatorname{Con}\mathbf{A}|_S$*.*
2. *The map $\theta \mapsto \theta_U$ is a $\wedge$- and $\vee$-preserving map from* $\operatorname{Con}\mathbf{A}$ *onto* $\operatorname{Con}\mathbf{A}|_U$*.*

Lemma 5.10. [13, proof of Lem. 8.3] *Let $\mathbf{A}$ be a finite algebra and let $\mathcal{L}_e(\mathbf{A})$ be the reduct of the language of $\mathbf{A}|_A$ which consists of those functions which satisfy $\mathbf{A}|_A \models \forall x : \ f(x, \ldots, x) = x$. Then for every set $T \subseteq A$ which is either a (δ, α)-minimal set or (δ, α)-trace for some cover pair (δ, α) of $\mathbf{A}$, there is an identification of $\mathcal{L}_e(\mathbf{A})$ with a reduct of the language of $\mathbf{A}|_T$ such that every* plain $\mathcal{L}_e(\mathbf{A})$-equation*,* *that is, an equation of the form* $f(x_{i_1}, \ldots, x_{i_n}) = g(x_{j_1}, \ldots, x_{j_m})$ *with* $f, g \in \mathcal{L}_e(\mathbf{A})$, *if universally true in* $\mathbf{A}|_A$, *is also universally true in* $\mathbf{A}|_T$.

To illustrate the application of the previous two lemmas, we sketch the proof of two technical results that will be cited in the next section. Note the interplay between algebras $\mathbf{B}$ and their induced polynomial algebras $\mathbf{B}|_S$ in the proofs.

Theorem 5.11. [13, Thm. 8.5] *Let* $\mathbf{A}$ *be a finite algebra. Then* $\mathbf{HSP}(\mathbf{A})$ *is congruence modular if and only if for every finite* $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ *and every cover pair* (δ, α) *of* $\mathbf{B}$,

(1) *typ*$(\delta, \alpha) \notin \{1, 5\}$.
(2) (δ, α)*-bodies coincide with* (δ, α)*-minimal sets.*

Sketch of proof. Assume first that $\mathbf{HSP}(\mathbf{A})$ is congruence modular. Then this fact is witnessed by the existence of a system of plain $\mathcal{L}_e(\mathbf{A})$-equations of a certain kind (known as *Day identities*) holding universally in $\mathbf{A}|_A$. Moreover, the functions occurring in the Day identities can be defined by terms in the language of $\mathbf{A}$ without parameters. Hence the universal validity of these Day identities is inherited by $\mathbf{B}|_B$ for every $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ and consequently (using Lemma 5.10) by $\mathbf{B}|_T$ whenever $\mathbf{B}$ is finite and T is a minimal set or trace for a cover pair of $\mathbf{B}$.

Now let $\mathbf{B}$ be a finite member of $\mathbf{HSP}(\mathbf{A})$, (δ, α) a cover pair of $\mathbf{B}$, U a (δ, α)-minimal set and N a (δ, α)-trace. We may factor by δ and thus assume $\delta = 0$. Then $\mathbf{B}|_N$ is completely known (Theorem 5.5) and cannot model Day identities in case its type is 1 or 5. A careful analysis shows that $\mathbf{B}|_U$ cannot model Day identities if U does not coincide with its body. This proves items (1) and (2).

Conversely, assume that for every finite $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ and every cover pair (δ, α) of $\mathbf{B}$, items (1) and (2) above hold. It suffices to prove that $\operatorname{Con}\mathbf{B}$ is a modular lattice for one particular *finite* member $\mathbf{B}$ of $\mathbf{HSP}(\mathbf{A})$. For each cover pair (δ, α) of $\mathbf{B}$ pick a (δ, α)-minimal set $U = U_{(\delta,\alpha)}$ and let $I = \{U_{(\delta,\alpha)} : (\delta, \alpha) \text{ a cover pair of } \mathbf{B}\}$. By items (1) and (2) and detailed knowledge of polynomial algebras induced on bodies of types 2–4, it can be shown that $\operatorname{Con}\mathbf{B}|_U$ is a modular lattice for each $U \in I$.

It can be deduced from Lemma 5.9 that the family of maps $\{\theta \mapsto \theta|_U : U \in I\}$ is a representation of the lattice $\operatorname{Con}\mathbf{B}$ by the modular lattices $\operatorname{Con}\mathbf{B}|_U$ $(U \in I)$. This implies $\operatorname{Con}\mathbf{B}$ itself is a modular lattice, as required. □

Lemma 5.12. [13, Lem. 6.5] *Suppose* $\mathbf{A}$ *is a finite algebra and* $0, \alpha, \beta, \theta, \mu$ *are distinct congruences forming a* pentagon sublattice *of* $\operatorname{Con}\mathbf{A}$ *as pictured in Figure 1. Assume* (α, β) *and* $(0, \varphi)$ *are cover pairs. If typ*$(\alpha, \beta) \neq 1$ *then typ*$(0, \varphi) \in \{3, 4, 5\}$.

Sketch of proof. Let U be a (α, β)-minimal set, B its body, and $T = U \setminus B$. The map $\theta \mapsto \theta|_B$ is a $\wedge$-preserving map from $\operatorname{Con}\mathbf{A}$ to $\operatorname{Con}\mathbf{A}|_B$ by Lemma 5.9, from which it follows that either $\varphi|_B = 0|_B$ or $\{\alpha|_B, \beta|_B, \varphi|_B\}$ generate a pentagon sublattice of

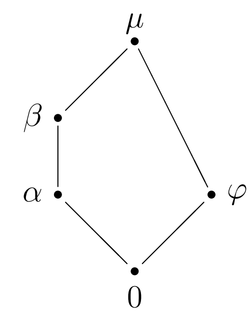

FIGURE 1. Pentagon sublattice

Con $\mathbf{A}|_B$. Using detailed knowledge of $\mathbf{A}|_B$ in types 2–5, one can show that the latter case is possible only if $\mathrm{typ}(\alpha,\beta)=1$.

So assume $\varphi|_B = 0|_B$. Because $\alpha|_B < \beta|_B$ and $\alpha|_T = \beta|_T$ (by definition of (α,β)-bodies) while $\beta_U \leq \alpha|_U \vee \beta|_U$ (by Lemma 5.9) and $\varphi|_B = 0|_B$ (by assumption), it follows that some block of $\varphi|_U$ is contained neither entirely in B nor entirely in T. Assume $\mathrm{typ}(0,\varphi) \notin \{3,4,5\}$. Then φ is abelian by Theorem 5.8, and this is inherited by $\varphi|_U$. Now a careful analysis of $\mathbf{A}|_U$ in types 2–5 shows that in those cases it cannot have a congruence $\varphi|_U$ with these two properties. Hence $\mathrm{typ}(\alpha,\beta)=1$ if $\mathrm{typ}(0,\varphi) \notin \{3,4,5\}$ in this case. □

5.7. **A sample application.** One of the early successes of tame congruence theory was the following application by Hobby and McKenzie to the RS conjectures.

Theorem 5.13. [13, Thm. 9.18, 10.4] *Let ε be any lattice equation which is nontrivial, i.e., fails to be true in some lattice. Then Theorem 4.2 is true with "ε" replacing "modularity." That is, the "Gap" and "Recursive" RS conjectures hold for those* $\mathbf{A}$ *for which* Con $\mathbf{B}$ *universally satisfies ε for every* $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$.

Sketch of proof. One can effectively determine from $\mathbf{A}$ whether $\mathbf{HSP}(\mathbf{A})$ is congruence modular. By Theorem 4.2, we may therefore assume that $\mathbf{HSP}(\mathbf{A})$ is not congruence modular. What will be shown in this case is that there is no bound to the cardinalities of the irreducibles in $\mathbf{HSP}(\mathbf{A})$.

The hypothesis that Con $\mathbf{B}$ satisfies ε for every $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ yields a system of plain $\mathcal{L}_e(\mathbf{A})$-equations holding universally in $\mathbf{A}|_A$ but which does not hold universally in any primitive permutation group as a G-set, the 2-element set with no functions, or the 2-element semilattice. As in the proof of Theorem 5.11, it follows that no cover pair for a finite algebra in $\mathbf{HSP}(\mathbf{A})$ can have type 1 or 5.

Because $\mathbf{HSP}(\mathbf{A})$ is not congruence modular, there is a finite algebra $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ for which Con $\mathbf{B}$ has a pentagon sublattice as in Figure 1. By finiteness of Con $\mathbf{B}$ we may assume that (α,β) is a cover pair. Moreover, we may assume that either $(0,\varphi)$ is a cover pair or the pentagon sublattice can be extended to a larger sublattice whose shape is known by the technical name "D_2." A lemma similar to Lemma 5.12 state that whenever $\operatorname{Con}\mathbf{B}$ has a D_2 sublattice, a particular cover pair in the convex closure of the D_2 sublattice must have type 1 or 5. Since we have established that $\mathbf{HSP}(\mathbf{A})$ omits types 1 and 5, it follows that we are in the former case, i.e., $(0,\varphi)$ can be assumed to be a cover pair. Then Lemma 5.12 implies that $\operatorname{typ}(0,\varphi) = 3$ or 4.

The argument now breaks into cases according to the type of (α,β). If $\operatorname{typ}(\alpha,\beta) = 2$ then in $\mathbf{B}$ we essentially have a 2-element lattice or boolean algebra (defined on a $(0,\varphi)$-trace) "independent from" a vector space (defined on an (α,β)-trace). A uniform construction similar in spirit to the Freese-McKenzie construction in the congruence modular case produces irreducibles in $\mathbf{HSP}(\mathbf{A})$ of every infinite cardinality. On the other hand, if $\operatorname{typ}(\alpha,\beta) = 3$ or 4 then the underlying geometry together with the "independence" of two polynomial algebras induced on type 3 or 4 (α,β)-minimal sets form the basis for a separate construction. □

Note that the role of the assumption about the lattice equation ε in the preceding proof was exactly to eliminate the presence of types 1 and 5 in $\mathbf{HSP}(\mathbf{A})$. Thus we have the following corollary of the proof.

Corollary 5.14. *The "Gap" and "Recursive" RS conjectures hold for all finite algebras* $\mathbf{A}$ *for which* $\mathbf{HSP}(\mathbf{A})$ *omits types 1 and 5.*

6. Universal algebra since 1985: a sampler

The deepest results in universal algebra in the last twenty years have depended almost without exception on the tools made available by tame congruence theory and modular commutator theory. This section contains a few of the highlights.

6.1. **Decidable first-order theories and good structure.** The *Decidable Equational Class Problem*, in its simplest version, asks the following two questions:

(1) For which finite algebras $\mathbf{A}$ in a finite language is the first-order theory of $\mathbf{HSP}(\mathbf{A})$ recursive?
(2) For which finite algebras $\mathbf{A}$ does the first-order theory of $\mathbf{HSP}(\mathbf{A})$ fail to interpret the theory of graphs?

In the late 1980's, McKenzie and M. Valeriote [33], building on similar work of S. Burris and McKenzie [4] in the congruence modular case, completely solved (2) and reduced (1) to the following question:

(1)′ For which finite rings R is the first-order theory of R-modules recursive?

Their solution uses tame congruence theoretic arguments that fill a book to prove the following.

Theorem 6.1. [33] *If* $\mathbf{A}$ *is a finite algebra, then the first-order theory of* $\mathbf{HSP}(\mathbf{A})$ *fails to interpret the theory of graphs if and only if* $\mathbf{HSP}(\mathbf{A})$ *itself decomposes in a particularly strong way as a "product"*

$$\mathbf{HSP}(\mathbf{A}) = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$$

where

(1) *For* $i = 1, 2, 3$, $\mathcal{V}_i$ *is an equational subclass of* $\mathbf{HSP}(\mathbf{A})$ *in which only type* i *occurs.*
(2) *Every member* $\mathbf{B} \in \mathbf{HSP}(\mathbf{A})$ *decomposes uniquely as a direct product* $\mathbf{B}_1 \times \mathbf{B}_2 \times \mathbf{B}_3$ *with* $\mathbf{B}_i \in \mathcal{V}_i$.
(3) *Every member of* $\mathcal{V}_1 \cup \mathcal{V}_2$ *is abelian.*
(4) $\mathcal{V}_3$ *is a* finitely generated discriminator variety *(a well-studied category of congruence distributive equational classes which includes, as typical examples, the class of boolean algebras and more generally, for each* $n \geq 2$, *the class* $\mathcal{CR}_n$ *of (commutative) rings satisfying* $\forall x : \ x^n = x$*; such classes are noteworthy for the fact that their members have topological representations over Stone spaces and recursive first-order theories).*
(5) $\mathcal{V}_2$ *is congruence modular and is definitionally equivalent in a strong sense to the class of* R*-modules for some finite ring* R.
(6) $\mathcal{V}_1$ *is definitionally equivalent to an equational class of* k*-sorted algebras all of whose functions have arity 1 and which satisfies one further condition.*

If in addition the language of $\mathbf{A}$ *is finite, then the first-order theory of* $\mathbf{HSP}(\mathbf{A})$ *is recursive if and only if all of the above items are true and moreover the first-order theory of* R*-modules is recursive, where* R *is the finite ring associated with* $\mathcal{V}_2$.

Using ideas contained in the proof of the previous theorem, B. Hart, S. Starchenko and Valeriote proved the following special case of Vaught's Conjecture (extended to incomplete theories).

Theorem 6.2. [12] *Any equational class* $\mathcal{V}$ *in a countable language has either countably many or continuum many countable models. Moreover, if the number of countable models is countable, then* $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ *with* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *satisfying descriptions related to those in Theorem 6.1.*

In fact, Hart, Starchenko and Valeriote prove such a decomposition assuming only that every member of $\mathcal{V}$ is superstable. Though they are not employing tame congruence theory, their techniques were inspired by the combinatorial arguments of tame congruence theory, replacing finiteness with the assumption of superstability.

An interesting variation of the decidable equational class problem is the *finitely decidable equational class problem*, which poses the original the pair of questions (recursive, noninterpretability of graphs) for the theory of the class of *finite* members of $\mathbf{HSP}(\mathbf{A})$. This variation was vigorously studied in the 1990's, culminating in the solution, due to J. Jeong [15] and P.M. Idziak [14], in the case that $\mathbf{HSP}(\mathbf{A})$ omits type 1 (the recursive question again being reduced to the case of R-modules for finite rings R). The solution gives characterizations of good structure for $\mathbf{HSP}(\mathbf{A})$ weaker and somewhat more complicated than the conditions given in Theorem 6.1; as in the proof of Theorem 6.1, the conditions are proved necessary by intricate tame congruence theoretic arguments, and sufficient by an appropriate Feferman-Vaught-style analysis.

Finally, J. Berman and Idziak have defined the notion of a finitely generated equational class $\mathbf{HSP}(\mathbf{A})$ having *few finite models*; by this they mean that the number of k-generated members of $\mathbf{HSP}(\mathbf{A})$ up to isomorphism is bounded by $2^{p(k)}$ for some polynomial $p(k)$. Using arguments inspired by and going beyond those in the study of the finitely decidable equational class problem, the monograph [3] of Berman and Idziak characterizes those $\mathbf{HSP}(\mathbf{A})$ which have few finite models and omit type 1.

6.2. **Refutation of the RS conjectures.** Before 1993, the best completely general result concerning the RS conjecture was the following theorem of McKenzie and Shelah, amplifying on an earlier result of W. Taylor.

Theorem 6.3. [31] *Let $\mathcal{V}$ be an equational class in a countable language. Then the cardinalities of the irreducible members of $\mathcal{V}$:*

(1) *are not bounded by any cardinal, or*
(2) *include all infinite cardinals less than or equal to $2^{\aleph_0}$ but no cardinal larger than $2^{\aleph_0}$, or*
(3) *include $\aleph_0$ but no cardinal larger than $\aleph_0$, or*
(4) *include arbitrarily large finite cardinals but no infinite cardinal, or*
(5) *are bounded by a finite cardinal.*

The RS "Gap" conjecture asserts that only the first and last options can arise if $\mathcal{V} = \mathbf{HSP}(\mathbf{A})$ for some finite algebra $\mathbf{A}$.

In the fall of 1993, while attempting to prove the relative version of the RS "Gap" conjecture for irreducible algebras with least nonzero congruence of type 5, McKenzie discovered a technique by which he could do roughly the following. Given a universal Horn theory T of directed graphs extending the universal Horn theory of some finite relational structure $\mathbf{M}$, McKenzie could build a finite algebra $\mathbf{A}$ with the property that all but finitely many of the irreducible members of $\mathbf{HSP}(\mathbf{A})$ (i) have a least nonzero congruence of type 5, and (ii) are bi-interpretable with those models of T which have a *sink vertex*, that is, a vertex which is reachable by paths from every vertex. Moreover, the language of $\mathbf{A}$ can be taken to be finite if T is finitely axiomatizable relative to the universal Horn theory of $\mathbf{M}$. If T is taken to be the theory of directed graphs asserting that the in- and out-degrees of every vertex are at most 1, then a suitable finite structure $\mathbf{M}$ exists, hence $\mathbf{A}$ can be built and $\mathbf{HSP}(\mathbf{A})$ has a unique countably infinite irreducible member but no uncountable irreducible members. This example refuted the RS "Gap" conjecture. By extensions of the technique to universal Horn theories of more complicated structures, McKenzie proved:

Theorem 6.4. [26] *For each* $k = 1, \ldots, 5$ *there exists a finite algebra* $\mathbf{A}_k$ *for which the cardinalities of the irreducible members of* $\mathbf{HSP}(\mathbf{A}_k)$ *are described by option* (k) *of Theorem 6.3.*

Except in the case $k = 4$, McKenzie also showed that it is possible to choose the algebra $\mathbf{A}_k$ witnessing the above theorem to have a finite language. The question of whether there $\mathbf{A}_4$ can be chosen with a finite language has been open since 1971 and is known as the *restricted Quackenbush problem.*

McKenzie also saw that, by the same technique, he could construct a universal Horn theory, finitely axiomatizable relative to the universal Horn theory of some finite structure, whose sink-connected models encode fragments of computations of Turing machines. In this way he proved the following response to the RS "Recursive" conjecture.

Theorem 6.5. [29] *For each* $k = 1, \ldots, 5$ *let* $\mathcal{C}_k$ *denote the class of (integer codes of) finite algebras* $\mathbf{A}$ *in a finite language such that the cardinalities of the irreducibles in* $\mathbf{HSP}(\mathbf{A})$ *are described by option (k) of Theorem 6.3. Then* $\mathcal{C}_1$ *and* $\mathcal{C}_5$ *are recursively inseparable.*

6.3. **Tarski's finite basis problem and other undecidable problems.** Recall from section 1:

Finite Basis Problem: For which finite algebras $\mathbf{A}$ with a finite language is $\mathbf{HSP}(\mathbf{A})$ finitely axiomatizable?

Tarski's Finite Basis Problem: Is there an algorithm which, given a finite algebra $\mathbf{A}$ in a finite language, decides whether the class $\mathbf{HSP}(\mathbf{A})$ is finitely axiomatizable?

In 1993 McKenzie proved that the answer to Tarski's question is "no." Here is how he did it. To prove an early version of Theorem 6.5, McKenzie gave an effective procedure which associates to each Turing machine $\mathcal{T}$ a finite algebra $\mathbf{A}(\mathcal{T})$ in such a way that:

(1) If $\mathcal{T}$ halts when started on a blank initial tape, then there is a finite bound to the cardinalities of the irreducible members of $\mathbf{HSP}(\mathbf{A}(\mathcal{T}))$.
(2) If $\mathcal{T}$ does not halt when started on a blank initial tape, then $\mathbf{HSP}(\mathbf{A}(\mathcal{T}))$ has countably infinite irreducible members.

Moreover, the language of $\mathbf{A}(\mathcal{T})$ is finite, and one of the countably infinite irreducibles in $\mathbf{HSP}(\mathbf{A}(\mathcal{T}))$ in the nonhalting case has properties known to imply that its presence in $\mathbf{HSP}(\mathbf{A}(\mathcal{T}))$ guarantees that $\mathbf{HSP}(\mathbf{A}(\mathcal{T}))$ is *not* finitely axiomatizable. Motivated by this, McKenzie gave another construction $\mathbf{F}(\mathcal{T})$ for which $\mathbf{HSP}(\mathbf{F}(\mathcal{T}))$

had the same countably infinite irreducible member in the nonhalting case and which he could show was finitely axiomatizable in the halting case. Thus McKenzie had reduced the halting problem to the finite basis problem, thereby answering Tarski's problem negatively.

Theorem 6.6. [28] *The answer to Tarski's finite basis problem is 'no': there is no algorithm which, given a finite algebra* $\mathbf{A}$ *in a finite language, determines whether* $\mathbf{HSP}(\mathbf{A})$ *is finitely axiomatizable.*

Since their publication, McKenzie's Theorems 6.5 and 6.6 have led to other undecidability results proved by his methods, including:

Theorem 6.7. *(McKenzie, J. Woods* [32]*) For each* $i \in \{2, 3, 4, 5\}$ *there is no algorithm which, given a finite algebra* $\mathbf{A}$ *in a finite language, determines whether type i occurs in* $\mathbf{HSP}(\mathbf{A})$.

Theorem 6.8. *(Willard* [39]*) There is no algorithm which, given a finite algebra* $\mathbf{A}$ *in a finite language, determines whether the first-order theory of* $\mathbf{HSP}(\mathbf{A})$ *has a model companion.*

6.4. **Park's conjecture and positive finite axiomatizability results.** Related to the finite basis problem is the following conjecture of R. Park from 1976.

Park's Conjecture 6.9. Let $\mathbf{A}$ be a finite algebra in a finite language. If there is a finite bound to the cardinalities of the irreducible members of $\mathbf{HSP}(\mathbf{A})$, then $\mathbf{HSP}(\mathbf{A})$ is finitely axiomatizable.

The impetus for this conjecture was primarily the celebrated finite basis theorem of K. Baker, which confirmed the conjecture in case $\mathbf{HSP}(\mathbf{A})$ is congruence distributive.

Theorem 6.10. [1] *Park's conjecture is true whenever* $\mathbf{HSP}(\mathbf{A})$ *is congruence distributive.*

(Note: by Theorem 3.3, if $\mathbf{HSP}(\mathbf{A})$ is congruence distributive then the cardinalities of the irreducibles in $\mathbf{HSP}(\mathbf{A})$ automatically have a finite bound.)

McKenzie subsequently verified Park's conjecture in the congruence modular case.

Theorem 6.11. [25] *Park's conjecture is true whenever* $\mathbf{HSP}(\mathbf{A})$ *is congruence modular.*

Interest in the conjecture grew significantly when McKenzie refuted the RS conjectures in 1993. From the properties stated above it is evident that his construction $\mathbf{A}(\mathcal{T})$ would also solve Tarski's problem if Park's conjecture could be verified for $\mathbf{A}(\mathcal{T})$ in the halting case. This was eventually done by me and led to further verifications of Park's conjecture.

Theorem 6.12. [38] *Park's conjecture is true whenever* $\mathbf{HSP}(\mathbf{A})$ *omits the abelian types 1 and 2.*

This last theorem has itself been generalized in two directions. On the one hand, Baker, G. McNulty and J. Wang [2] have succeeded in weakening the hypothesis that there be a finite bound on the cardinalities of the irreducible elements in $\mathbf{HSP}(\mathbf{A})$. In a different direction, M. Maróti and McKenzie [24] have generalized the result to a significant class of universal Horn classes generated by finitely many finite algebras.

6.5. **The RS problem, nilpotency, and variations of abelianness.** McKenzie's refutation of the RS "Gap" and "Recursive" conjectures, rather than ending efforts to understand the nature of irreducibles in finitely generated equational classes, instead served to sharpen the focus of the small group of specialists engaged in this analysis. The following somewhat ill-defined problem and its accompanying concrete subproblem describe in part the ongoing aim of these specialists.

RS Problem 6.13.

(1) (Structure) Characterize those finitely generated equational classes $\mathbf{HSP}(\mathbf{A})$ ($\mathbf{A}$ finite) for which there is a cardinal bound to the cardinalities of its irreducible members.
(2) (co-R.E.) Prove that the property above is a co-r.e. property of $\mathbf{A}$.

By the way, it is now time to reveal the secret of the acronym "RS." It stands for "residually small," jargon for "having a cardinal bound to the cardinalities of its irreducible members."

Freese's and McKenzie's proof of Theorem 4.2 completely solved both parts of the RS Problem in the congruence modular case. The first paragraph of the proof of Theorem 5.13, together with the observation preceding Corollary 5.14, imply that the problem is solved in case types 1 and 5 are omitted.

Theorem 6.14. *The RS "Structure" problem is solved and the "co-R.E." problem is confirmed for all finite algebras* $\mathbf{A}$ *for which* $\mathbf{HSP}(\mathbf{A})$ *omits types 1 and 5.*

McKenzie refuted the RS conjectures with finitely generated equational classes which omitted the abelian types 1 and 2. Notwithstanding this fact, McKenzie had previously solved the RS *problems* in that case. Moreover, if one relativizes the RS problem to the consideration of only those irreducibles whose least nonzero congruence is nonabelian, then McKenzie had completely solved this relativized problem.

Theorem 6.15. [30] *When the RS problem is relativized to those irreducible members of* $\mathbf{HSP}(\mathbf{A})$ *with least nonzero congruence of types 3–5, the "Structure" problem is solved and the "co-R.E." problem is confirmed.*

Subsequently, K. Kearnes and McKenzie in unpublished work have independently solved the RS problems relativized to type 2.

Theorem 6.16. [17] *When the RS problem is relativized to those irreducible members of* $\mathbf{HSP}(\mathbf{A})$ *with least nonzero congruence of type 2, the "Structure" problem is solved and the "co-R.E." problem is confirmed.*

Therefore in a sense the only work remaining is the study of irreducibles whose least nonzero congruence has type 1. This statement obscures the magnitude of the problem, however; such irreducibles live outside "classical" mathematics, and their understanding seems to require entirely new tools. The situation is easiest to describe when considering equational classes $\mathbf{HSP}(\mathbf{A})$ which admit only the abelian types 1 and 2. Finite algebras in such classes are *solvable* in a sense analogous to the notion of solvable groups. Also by analogy it is possible to define the concept of one congruence *centralizing* another, so that a congruence is abelian if and only if it centralizes itself. With this notion of centrality one can define usual notions of nilpotency, but unlike the situation in groups the notions are not equivalent, and in general are not well-behaved or understood. In 1993 Kearnes published a ground-breaking paper [16] that uncovered deep consequences of these notions of nilpotency in finite algebras, proving, for example, that every homomorphic image of a *finite* abelian algebra is nilpotent in all senses.

Building on Kearnes' earlier work, Kearnes, E. Kiss and Valeriote in [20] developed a new theory of higher-dimensional traces, or *multitraces*, for cover pairs of types 1–3, and exploited this theory in [21] to resolve the RS problems in the case that every member of $\mathbf{HSP}(\mathbf{A})$ is abelian. In fact they prove:

Theorem 6.17. *The RS "Structure" problem is solved and the RS conjectures are confirmed for all finite algebras* $\mathbf{A}$ *for which every member of* $\mathbf{HSP}(\mathbf{A})$ *is abelian.*

Tame congruence theory contained from the beginning a second notion of abelianness, called *strong abelianness*, which separates G-sets from R-modules (the former are, the latter are not). Kearnes and Kiss in [18] introduced a strong centrality relation which captures strong abelianness, and teased from it two further notions of centrality (*weak* and *rectangular*), which in turn give rise to their own notions of abelianness. As well, they solve the RS problems for finite strongly nilpotent algebras:

Theorem 6.18. *The RS "Structure" problem is solved and the RS conjectures are confirmed for all finite strongly nilpotent algebras* $\mathbf{A}$.

Kearnes and Kiss then considered the more general case of nilpotent algebras. In [19] the authors define the notion of the *twin group* of a $(0, \alpha)$-trace and relate its properties to weak abelianness and nilpotence. In particular, they prove that if $\mathbf{A}$ is a finite nilpotent algebra, then a necessary consequence of $\mathbf{HSP}(\mathbf{A})$ being residually small is that the twin groups which arise must be abelian and hence every member of $\mathbf{HSP}(\mathbf{A})$ must be weakly abelian. As a corollary they are able to solve the RS "Structure" problem and verify the RS conjectures for a special kind of nilpotent algebra which arises naturally in tame congruence theory. And here rests the current state of the RS problems.

7. Further reading

For an introduction to universal algebra see the classic text of S. Burris and H.P. Sankappanavar [5], which is out of print but can be downloaded for free (see the reference). Modular commutator theory is thoroughly covered in [8]; a slimmer introduction is [9]. The authoritative text for tame congruence theory is [13], which can be downloaded from the American Mathematical Society. The reader might wish to consult the introductions to (parts of) tame congruence theory given by [6] and [23]. The chapters [22, 36, 37] from [11] give more detailed surveys of parts of this article.

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