AN ALGEBRA THAT IS DUALIZABLE BUT NOT FULLY DUALIZABLE

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Abstract. We give an example of a finite algebra which is dualizable but not fully dualizable in the sense of natural duality theory.

1. Introduction

Fix a finite algebra $M$, in the sense of universal algebra, and let $M$ be an infinitary topological structure whose universe is identical to that of $M$, whose topology is discrete, and whose signature consists only of $\lambda$-ary relations ($\lambda$ a nonzero ordinal) which are universes of subalgebras of $M^\lambda$, and $\lambda$-ary operations or partial operations ($\lambda$ an ordinal) whose graphs are universes of subalgebras of $M^{\lambda+1}$. (Such a structure is called an alter ego of $M$ in [1].) The quasivariety $\mathcal{A}$ generated by $M$ and the topological quasivariety $\mathcal{X}$ generated by $M$ are dually adjoint via the functors $D = \text{Hom}(-, M)$ and $E = \text{Hom}(-, M)$ and the “evaluation map” natural transformations $e : 1_{\mathcal{A}} \to ED$ and $\varepsilon : 1_{\mathcal{X}} \to DE$. The transformation $e$ is defined as follows: given $A \in \mathcal{A}$ and $a \in A$, define $e^A(a) : D(A) \to A$ by $e^A(a)(h) = h(a)$; then $e^A : A \mapsto E(D(A))$ is the map $a \mapsto e^A(a)$. The definition of $\varepsilon$ is similar.

$M$ yields a duality on $M$ if $e^A$ is an isomorphism for every $A \in \mathcal{A}$, and yields a full duality on $M$ if in addition $\varepsilon^X$ is an isomorphism for every $X \in \mathcal{X}$. Thus if $M$ yields a full duality on $M$, then the above dual adjunction provides a “natural” dual equivalence between $\mathcal{A}$ and a finitely generated topological quasivariety.

If the signature of $M$ consists of the proper class of all permissible relations, operations and partial operations, then $M$ automatically yields a full duality on $M$, but we consider this to be cheating. Following [3], we say that $M$ is dualizable if there exists an alter ego which yields a duality on $M$ and whose signature consists of finitary relations, operations and partial operations only. $M$ is fully dualizable if there exists such an alter ego which yields a full duality on $M$. Not all finite algebras are fully dualizable or even dualizable in this sense; however, in 1991 B. A. Davey noted that every algebra known to be dualizable had been shown to be fully dualizable, and asked [3, Problem 4] whether the two notions are equivalent. We give a

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strong negative answer to this question by displaying a 3-element algebra which is dualizable but is not fully dualized by any alter ego having only a set of (possibly infinitary) relations, operations and partial operations in its signature. This also solves the “Strong Upgrade Problem” in [1].

2. $M$ is dualizable

Our algebra is $M = \langle M, f, g \rangle$ where $M = \{0, 1, 2\}$ and $f, g$ are the unary operations defined by the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Consider $\langle M, \leq \rangle$ as an ordered chain with $0 < 1 < 2$. Let $\wedge$ and $\vee$ denote the lattice meet and join operations in this chain. Define $E \subseteq M^2$ and $R \subseteq M^4$ as follows:

$E = \{(x, y) : x \leq y \text{ and } (x, y) \neq (0, 2)\}$

$R = \{(x, y, z, w) : x \leq y \leq z \leq w \text{ and } x = y \text{ or } z = w\}$.

Define $\mathbb{M} = \langle M, \wedge, \vee, E, R, \text{ discrete topology} \rangle$. It can be easily checked that $E$ is a subuniverse of $M^2$, the graphs of $\wedge$ and $\vee$ are subuniverses of $M^3$, and $R$ is a subuniverse of $M^4$; in other words, that $\mathbb{M}$ is a finitary alter ego for $M$.

**Theorem 2.1.** $\mathbb{M}$ dualizes $M$.

**Proof.** We will show that $\mathbb{M}$ satisfies the so-called Interpolation Condition relative to $M$. The result then follows either by the Second Duality Theorem [1, Theorem 2.2.7], or by [1, Lemma 2.2.5] and the Duality Compactness Theorem of L. Zádori (see [6, Corollary 3.5] or [1, Theorem 2.2.11]).

To say that $\mathbb{M}$ satisfies the Interpolation Condition relative to $M$ is just to say that if $1 \leq n < \omega$ and $X \leq M^n$ and $h \in \text{Hom}(X, \mathbb{M})$, then $h$ is the restriction to $X$ of an $n$-ary term operation of $M$. The proof is by cases.

**Case 1.** $|\text{range}(h)| = 1$.

That is, $h$ is constant. Then $h$ is the restriction to $X$ of either $gg(x_1)$, $fgg(x_1)$ or $ff(x_1)$.

**Case 2.** $|\text{range}(h)| = 2$.

Then $\langle X, \wedge, \vee \rangle$ is a finite distributive lattice and $h$ is a homomorphism from this lattice onto a two-element lattice. Write $\text{range}(h) = \{0', 1'\}$ with $0' < 1'$. Thus there exist $a, b \in X$ such that $h^{-1}(0') = \{x \in X : x \leq a\}$ and $h^{-1}(1') = \{x \in X : x \geq b\}$. Let $a^* = a \vee b$.

Suppose first that $\text{range}(h) = \{0, 2\}$. Since $\langle h(a), h(a^*) \rangle \notin E$ and $h$ preserves $E$, there must exist $i \in \{1, \ldots, n\}$ such that $(a_i, a_i^*) \notin E$. The only way this can happen
is if \((a_i, a_i^*) = (0, 2)\). Since \(a_i^* = a_i \lor b_i\) and \(\langle M, \land, \lor \rangle\) is a chain, we get \(b_i = 2\). It follows that for all \(x \in X\), \(h(x) = 0\) implies \(x_i = 0\) while \(h(x) = 2\) implies \(x_i = 2\). Thus \(h\) is the restriction to \(X\) of the coordinate projection \(x_i\).

Suppose on the other hand that \(1 \in \text{range}(h)\). Choose any \(i \in \{1, \ldots, n\}\) such that \(a_i < a_i^*\). Then as in the previous paragraph we get \(a_i < b_i\). Clearly if \(x, y \in X\) and \(x_i = y_i\), then it is impossible to have \(x \leq a\) while \(y \geq b\), or vice versa. Thus \(x_i = y_i\) implies \(h(x) = h(y)\). This means that if \(X_i = \{x_i : x \in X\}\) then we can define \(h_i : X_i \rightarrow \text{range}(h)\) so that \(h(x) = h_i(x_i)\) for all \(x \in X\). Our goal is now to prove that \(h_i\) is the restriction to \(X_i\) of a unary term operation \(t\) of \(M\), for then \(h\) will be \(t(x_i)|_X\). Since \(1 \in \text{range}(h_i)\), it suffices to prove that \(h_i\) is order-preserving. If \(x, y \in X\) with \(x_i \leq y_i\), then
\[
h_i(x_i) = h_i((x \land y)_i) = h(x \land y) = h(x) \land h(y) = h_i(x_i) \land h_i(y_i),
\]
so \(h_i(x_i) \leq h_i(y_i)\) as required.

**Case 3.** \(|\text{range}(h)| = 3\).

That is, \(h\) is surjective. Then \(h\) is a lattice homomorphism from \(\langle X, \land, \lor \rangle\) onto a three-element chain. Thus there exist \(a, b, c, d \in X\) such that
\[
\begin{align*}
h^{-1}(0) & = \{x \in X : x \leq a\} \\
h^{-1}(1) & = \{x \in X : b \leq x \leq c\} \\
h^{-1}(2) & = \{x \in X : d \leq x\}.
\end{align*}
\]
Let \(a^* = a \lor b\) and \(c^* = c \lor d\), and note that \(a^* \leq c\) (as \(h(a^*) = h(a) \lor h(b) = 1\)).

Since \(\langle h(a), h(a^*), h(c), h(c^*) \rangle \notin R\), and since \(h\) preserves \(R\), there must exist \(i\) such that \((a_i, a_i^*, c_i, c_i^*) \notin R\). This forces \((a_i, a_i^*, c_i, c_i^*) = (0, 1, 1, 2)\), hence \(b_i = 1\) and \(d_i = 2\). It follows that \(h\) is the restriction to \(X\) of the coordinate projection \(x_i\). \(\square\)

### 3. Reduction to a simpler category

By a *looped dag* we mean a directed graph \(G = \langle G, \rightarrow \rangle\) in which the vertex set \(G\) is possibly empty, the edge relation \(\rightarrow \subseteq G \times G\) possibly has loops \(a \rightarrow a\), and there do not exist *distinct* vertices \(a_0, a_1, \ldots, a_n, n \geq 1\), such that \(a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n \rightarrow a_0\). \(D\) denotes the class of all looped dags.

If \(G \in D\), then we canonically define a larger looped dag \(G^+ = \langle G^+, \xrightarrow{\pm} \rangle\) as follows: let \(G^+\) be the disjoint union of \(G\) and the set \(\{0, 1\}\); then for \(x, y \in G^+\) define \(x \xrightarrow{\pm} y\) iff \(x = 0\) or \(y = 1\) or \(x \rightarrow y\). Also define \(G^+\) to be the structure \(\langle G^+, \xrightarrow{\pm}, 0, 1 \rangle\) and \(D^+ = \{G^+ : G \in D\}\). We consider \(D^+\) as a category with the usual homomorphisms. It turns out that \(\text{ISP}(M)\) is categorically equivalent to \(D^+\); we shall prove most of this statement. We use \(D\) to efficiently name the members of \(D^+\).
Lemma 3.1.

If \( G \in \mathcal{D} \), we wish to define an algebra \( B(G^+) \) in the language of \( M \). Its universe \( B(G^+) \) is the edge set \( \rightarrow \) of \( G^+ \). The operations \( f, g \) are defined on \( B(G^+) \) as follows:

\[
\begin{align*}
  f(x, y) &= (y, 1) \\
  g(x, y) &= (0, x).
\end{align*}
\]

Note in particular that if \( \emptyset \) is the empty looped dag, then \( \emptyset^+ = 2_{01} \), the 2-element bounded poset, and \( B(\emptyset^+) \cong M \) via the isomorphism \( \iota \) sending \((0, 0) \mapsto 0, (0, 1) \mapsto 1, \) and \((1, 1) \mapsto 2 \).

Given \( G, H \in \mathcal{D} \) and \( h \in \text{Hom}(G^+, H^+) \), define the map \( \beta_{G^+}(h) : B(G^+) \to B(H^+) \) by applying \( h \) coordinatewise: \( \beta_{G^+}(h)(x, y) = (h(x), h(y)) \).

**Lemma 3.1.**

1. The maps \( G^+ \hookrightarrow B(G^+), h \mapsto \beta_{G^+}(h) \), define a functor from \( \mathcal{D} \) to \( \text{ISP}(M) \).

2. For all \( G, H \in \mathcal{D} \), \( \text{Hom}(B(G^+), B(H^+)) = \{ \beta_{G^+}(h) : h \in \text{Hom}(G^+, H^+) \} \).

3. For any \( G \in \mathcal{D} \), the map \( h \mapsto \iota \beta_{G^+}(h) \) is an order isomorphism from \( (\text{Hom}(G^+, 2_{01}), \leq) \) to \( (\text{Hom}(B(G^+), M), \leq) \). (Here \( \leq \) denotes the order relations evaluated pointwise in \( 2 \) and \( M \) respectively.)

**Proof.** The first item will be clear once it is seen that \( B(G^+) \in \text{ISP}(M) \) when \( G \in \mathcal{D} \), and \( \beta_{G^+}(h) \in \text{Hom}(B(G^+), B(H^+)) \) when \( h \in \text{Hom}(G^+, H^+) \). Let \( < \) be the reflexive, transitive closure of \( \rightarrow \) on \( G^+ \). \( < \) is a partial order so there is some set \( I \) and an embedding of \( (G^+, <, 0, 1) \) into the bounded poset \( (2_{01})^I \). Using this embedding to relabel the vertices of \( G^+ \), we may assume that \( (G^+, <, 0, 1) \leq (2_{01})^I \).

Now define an embedding \( \tau : B(G^+) \hookrightarrow M^I \) as follows. For \( (x, y) \in B(G^+) \), we have \( x \leq y \) in \( (2_{01})^I \) and we define \( \tau(x, y) \) by

\[
\tau(x, y)_i = \begin{cases} 
  0 & \text{if } x_i = y_i = 0 \\
  1 & \text{if } x_i = 0, y_i = 1 \\
  2 & \text{if } x_i = y_i = 1.
\end{cases}
\]

The following shows that \( \tau \) preserves \( f \):

\[
f(\tau(x, y))_i = \begin{cases} 
  1 & \text{if } y_i = 0 \\
  2 & \text{if } y_i = 1
\end{cases} = \tau(y, 1)_i = \tau(f(x, y))_i.
\]

Similarly, \( \tau \) preserves \( g \). Clearly \( \tau \) is injective. Thus \( B(G^+) \in \text{ISP}(M) \).

Next, assume \( h \in \text{Hom}(G^+, H^+) \). If \( (x, y) \in B(G^+) \) then \( (x, y) \) is an edge of \( G^+ \), hence \( (h(x), h(y)) \) is an edge of \( H^+ \) and is an element of \( B(H^+) \). Let \( (x, y) \in B(G^+) \). Since \( h(0) = 0 \) and \( h(1) = 1 \) we have

\[
\beta_{G^+}(h)(g(x, y)) = \beta_{G^+}(h)(0, x) = (h(0), h(x)) = (0, h(x)) = g(h(x), h(y)) = g(\beta_{G^+}(h)(x, y)).
\]

Similarly, \( \beta_{G^+}(h)(f(x, y)) = f(\beta_{G^+}(h)(x, y)) \). Thus \( \beta_{G^+}(h) \in \text{Hom}(B(G^+), B(H^+)) \).
To prove the second item, let \( \alpha \in \text{Hom}(B(G^+), B(H^+)) \). Note that \( \alpha(0, 0) = (0, 0) \) and \( \alpha(0, 1) = (0, 1) \), since in both \( B(G^+) \) and \( B(H^+) \), \( (0, 0) \) is the unique element in the range of \( gg \) and \( (0, 1) = f(0, 0) \). Furthermore, in either \( B(G^+) \) or \( B(H^+) \) we have \( g(x, y) = (0, 0) \) iff \( x = 0 \). Thus there exists a function \( h : G^+ \to H^+ \) such that \( \alpha(0, y) = (0, h(y)) \) for all \( y \in H^+ \). Note in particular that \( h(0) = 0 \) and \( h(1) = 1 \).

Assume \( x \mapsto y \) in \( G^+ \); then \( (x, y) \in B(G^+) \) with, say, \( \alpha(x, y) = s \in B(H^+) \). Then \( g(s) = \alpha(0, x) = (0, h(x)) \) and \( gf(s) = g(\alpha(y, 1)) = \alpha(0, y) = (0, h(y)) \). By checking the definition of \( B(H^+) \) we see that this forces \( s = (h(x), h(y)) \), hence \( (h(x), h(y)) \in B(H^+) \) and therefore \( h(x) \mapsto h(y) \) in \( H^+ \). This proves \( h \in \text{Hom}(G^+, H^+) \).

We claim that whenever \( \alpha, \alpha' \in \text{Hom}(B(G^+), B(H^+)) \) and for all \( y \in G^+ \), \( \alpha(0, y) = \alpha'(0, y) \), then \( \alpha = \alpha' \). Indeed, let \( (x, y) \in B(G^+) \).

\[
\begin{align*}
f(\alpha(x, y)) &= \alpha(y, 1) = \alpha(f(0, y)) = f(\alpha(0, y)) = f(\alpha'(0, y)) = \alpha'(f(0, y)) \\
&= \alpha'(y, 1) = \alpha'(f(x, y)) = f(\alpha'(x, y)).
\end{align*}
\]

That \( g(\alpha(x, y)) = g(\alpha'(x, y)) \) is a similar but simpler calculation. In \( B(H^+) \), \( f(u) = f(v) \) and \( g(u) = g(v) \) imply \( u = v \) so we have \( \alpha = \alpha' \), proving the claim. Since \( \beta_{G^+}(h)(0, y) = (h(0), h(y)) = (0, h(y)) = \alpha(0, y) \), the claim yields \( \beta_{G^+}(h) = \alpha \).

To prove the third item, let \( \rho : \text{Hom}(G^+, 2_{01}) \to \text{Hom}(B(G^+), B(2_{01})) \) be given by \( h \mapsto \beta_{G^+}(h) \). \( \rho \) is surjective by item 2. It will suffice to show that \( \rho \) is an order embedding where the order in \( B(2_{01}) \) is the one inherited from \( M \) via \( \iota \). Assume \( \beta_{G^+}(h_1) = \beta_{G^+}(h_2) \). For all \( y \in G^+ \), \( (0, y) \in B(G^+) \) so \( \beta_{G^+}(h_1)(0, y) = \beta_{G^+}(h_2)(0, y) \) or \( h_1(y) = h_2(y) \). Hence \( \rho \) is injective.

If \( h_1, h_2 \in \text{Hom}(G^+, 2_{01}) \) with \( h_1 \leq h_2 \) then for all \( (x, y) \in B(G^+) \), \( (h_1(x), h_1(y)) \leq (h_2(x), h_2(y)) \) co-ordinatwise and hence in \( B(2_{01}) \). That is, \( \beta_{G^+}(h_1) \leq \beta_{G^+}(h_2) \).

Conversely, assume \( h_1 \not\leq h_2 \); choose \( y \in G^+ \) such that \( h_1(y) = 1 \) while \( h_2(y) = 0 \). Then \( \beta_{G^+}(h_1)(0, y) \not\leq \beta_{G^+}(h_2)(0, y) \), proving \( \beta_{G^+}(h_1) \not\leq \beta_{G^+}(h_2) \).

Given \( A \in \text{ISP}(M) \), we shall define a directed graph \( G_A^+ \) as follows. The vertex set is \( g(A) \), the range of \( g \) in \( A \). The edge set consists of a directed edge \( e_a = (g(a), gf(a)) \) for each \( a \in A \). Note that if 0 denotes the unique element in the range of \( gg \) in \( A \) and 1 denotes \( f(0) \), then \( 0, 1 \in g(A) \). Define \( G_A^+ = (G_A', 0, 1) \). Note that \( G_M^+ = 2_{01} \) and, more generally, \( G_{M^X}^+ = (2_{01})^X \) for any set \( X \).

**Lemma 3.2.**

1. The maps \( A \mapsto G_A^+ \) (for \( A \in \text{ISP}(M) \)) and \( h \mapsto h|_{g(A)} \) (for \( h \in \text{Hom}(A, B) \)) define a functor from \( \text{ISP}(M) \) to \( D^+ \).
2. For each \( A \in \text{ISP}(M) \), the map \( \eta_A : A \to B(G_A^+) \) defined by \( \eta_A(a) = e_a \) is an isomorphism from \( A \) to \( B(G_A^+) \).
3. \( \text{ISP}(M) = \text{I}(\{B(G^+) : G \in D\}) \).
4. If \( G \in D, A \in \text{ISP}(M), h \in \text{Hom}(G, G_A^+) \), and \( y \in G^+ \), then \( \beta_{G^+}(h)(0, y) = \eta_A(h(y)) \).
Lemma 3.5. For every algebraic operation following implication holds: if \( h \in \text{Hom}(A, B) \), then \( h|_{g(A)} \) is easily shown to be in \( \text{Hom}(G_A^+, G_B^+) \). In particular, an embedding of \( A \) into \( M^X \) induces an injective homomorphism from \( G_A^+ \) to \((2^{01})^X \). It follows that \( G'_A \) is a looped dag. Since \( gg(x) = 0 \) and \( gfg = g \) in \( A \), it follows that if \( a \in g(A) \) then \( e_a = (0, a) \) while \( e_f(a) = (a, 1) \). Thus in \( G'_A \) there is an edge from 0 to every vertex, and from every vertex to 1. Hence \( G_A^+ \in D^+ \). This proves the nontrivial parts of item 1.

The quasi-identity \( [g(x) = g(y) & gf(x) = gf(y)] \Rightarrow x = y \) is true in \( M \) and hence holds throughout \( ISP(M) \). Thus the map \( \eta_A : A \rightarrow B(G_A^+) \) is injective and hence is a bijection. To prove it is an isomorphism, observe that

\[
\begin{align*}
e_g(a) &= (gg(a), gfg(a)) = (0, g(a)) = g(e_a) \\
e_f(a) &= (gf(a), gff(a)) = (gf(a), 1) = f(e_a).
\end{align*}
\]

This proves item 2. The third item follows from item 2 and Lemma 3.1. The fourth item follows from comments in the first paragraph and the fact that \( h(y) \in g(A) \). □

An (infinitary) algebraic operation of \( M \) is any function \( \alpha \) such that for some nonzero ordinal \( \lambda \) and some \( D \leq M^\lambda \) we have \( \alpha \in \text{Hom}(D, M) \). The ordinal \( \lambda \) is the arity of \( \alpha \). Fix an algebraic operation \( \alpha \) of \( M \) (say \( \alpha \in \text{Hom}(D, M) \) with \( D \leq M^\lambda \)), and for each \( i < \lambda \) let \( \rho_i : D \rightarrow M \) denote the \( i \)th projection. If \( A \in ISP(M) \) and \( X \subseteq \text{Hom}(A, M) \), then \( X \) is said to be closed under \( \alpha \) if for all \( \delta \in \text{Hom}(A, D) \) the following implication holds: if \( \rho_i \circ \delta \in X \) for every \( i < \lambda \), then \( \alpha \circ \delta \in X \).

We wish to define the appropriate analogous notions for \( 2_{01} \).

Definition 3.3. An algebraic operation of \( 2_{01} \) is a pair \((H^+, h)\) where \( H^+ \subseteq D^+ \) for some nonzero ordinal \( \lambda \) and the inclusion map \( H^+ \hookrightarrow 2^\lambda \) is a homomorphism from \( H^+ \) to \((2^{01})^\lambda \), and where \( h \in \text{Hom}(H^+, 2_{01}) \). The ordinal \( \lambda \) is the arity of \((H^+, h)\).

Note that in general, the edge set of \( H^+ \) cannot be recovered from the graph of \( h \), hence the explicit inclusion of \( H^+ \) in the definition.

Definition 3.4. Suppose \( G \in D \), \( Y \subseteq \text{Hom}(G^+, 2_{01}) \), and \((H^+, h)\) is a \( \lambda \)-ary algebraic operation of \( 2_{01} \). For each \( i < \lambda \) let \( r_i : H^+ \rightarrow 2_{01} \) denote the \( i \)th projection. We say that \( Y \) is closed under \((H^+, h)\) relative to \( G^+ \) if for all \( d \in \text{Hom}(G^+, H^+) \) the following implication holds: if \( r_i \circ d \in Y \) for all \( i < \lambda \), then \( h \circ d \in Y \).

Lemma 3.5. For every algebraic operation \( \alpha \) of \( M \) there is an algebraic operation \((H^+, h)\) of \( 2_{01} \) of the same arity as \( \alpha \) making the following true: for any \( G \in D \) and any \( Y \subseteq \text{Hom}(G^+, 2_{01}) \), if \( Y \) is closed under \((H^+, h)\) relative to \( G^+ \), then the set \( X := \{ \iota \beta_{G^+}(y) : y \in Y \} \subseteq \text{Hom}(B(G^+), M) \) is closed under \( \alpha \).

Proof. Let \( D \leq M^\lambda \) and \( \alpha \in \text{Hom}(D, M) \) be given, and put \( H^+ = G_D^+ \). Then the inclusion map \( H^+ \hookrightarrow (2_{01})^\lambda \) is a homomorphism. Let \( \alpha^* = \iota^{-1} \alpha \eta_D^{-1} \in \text{Hom}(B(H^+), B(2_{01})) \). By Lemma 3.1(2) there exists \( h \in \text{Hom}(H^+, 2_{01}) \) such that \( \alpha^* = \beta_{H^+}(h) \). Thus
(H⁺, h) is an algebraic operation of 2₀₁ of the same arity as α; we shall show that it
witnesses the claim of Lemma 3.5.

Let G ∈ D and Y ⊆ Hom(G⁺, 2₀₁), and assume that Y is closed under (H⁺, h) relative to G⁺. Define X = {ιβ⁺(y) : y ∈ Y}, and let δ ∈ Hom(B(G⁺), D) be such that ρ₁ ∘ δ ∈ X for all i < λ; we must show α ∘ δ ∈ X.

By Lemma 3.1(2) there exists d ∈ Hom(G⁺, H⁺) such that ηDδ = β⁺G⁺(d). For each i < λ choose yi ∈ Y such that ρ₁ ∘ δ = ιβ⁺G⁺(yi). We first verify that ri ∘ d = yi for all i < λ.

Fix a ∈ G⁺ and define s = δ(0, a) ∈ D and t = δ(a, 1) ∈ D. As g(t) = s in D, i.e., s ∈ g(D), we have that s is simultaneously an element of D and of H⁺, and hence ρ₁(s) = ri(s) for all i < λ. Moreover, it follows from remarks in the proof of Lemma 3.2(1) that ηD(s) = (0, s), as s ∈ g(D). On the other hand, recall from the proof of Lemma 3.1(2) that ηD(s) = β⁺G⁺(d)(0, a) = (0, d(a)); hence s = d(a).

Finally, note that since 2₀₁ = G⁺ and using Lemma 3.2(4), for each i < λ we have β⁺G⁺(yi)(0, a) = ηMηM(yi(a)); hence ιβ⁺G⁺(yi)(0, a) = ι(yi(a)) (note that ι = ηM⁻¹). Thus (ri ∘ d)(a) = ri(s) = ρ₁(s) = (ρ₁ ∘ δ)(0, a) = iβ⁺G⁺(yi)(0, a) = yi(a), proving ri ∘ d = yi as desired.

In particular, ri ∘ d ∈ Y for all i < λ. As Y is closed under (H⁺, h) relative to G⁺, the map h ∘ d is in Y and hence ιβ⁺G⁺(h ∘ d) ∈ X. But ιβ⁺G⁺(h ∘ d) = ιβ⁺H⁺(h)β⁺G⁺(d) = iα⁺ηDδ = αδ, which proves α ∘ δ ∈ X as required.

4. M IS NOT FULLY DUALIZABLE

**Lemma 4.1.** For every infinite cardinal κ there exists a structure G⁺κ = ⟨G⁺κ, →, →⟩ satisfying:

1. → is a linear ordering of G⁺κ; → is a partial ordering of G⁺κ.
2. → is a proper subset of →.
3. For all x, y ∈ G⁺κ with x → y but x ↛ y, there exists {α₁ : i < κ} ∪ {β₁ : i < κ} ⊆ G⁺κ such that for all i < κ, x → α₁ → β₁ → y and β₁ → α₁⁺ → α₁⁺⁺.

**Proof.** It suffices to note that if G = ⟨G, →, →⟩ is a model of items 1 and 2 and has cardinality κ, and if (x, y) ∈ G² is a single failure of item 3, then G can be embedded in a larger model H of items 1 and 2, still of cardinality κ and in which the designated instance of item 3 is now true. Then a chain of models of 1 and 2 can be arranged so that its union is a model of all three items. □

For each infinite cardinal κ fix a structure G⁺κ = ⟨G⁺κ, →, →⟩ as in the previous lemma. Put G⁺κ = ⟨G⁺κ, →⟩ and L⁺κ = ⟨G⁺κ, →⟩. Define Y⁺κ = Hom(L⁺κ, 2₀₁) and note that Y⁺κ ⊆ Hom(G⁺κ, 2₀₁).

**Lemma 4.2.** Y⁺κ is closed relative to G⁺κ under all algebraic operations of 2₀₁ of arity less than κ.
Proof. Let \((H^+, h)\) be an algebraic operation of \(2^\lambda_0\) where \(H^+ \subseteq 2^\lambda\) with \(\lambda < \kappa\).

Suppose \(d \in \text{Hom}(G^+_\kappa, H^+)\) is such that \(r_i \circ d \in Y_\kappa\) for all \(i < \lambda\). Let \(\nu : H^+ \hookrightarrow 2^\lambda\) denote the inclusion map. Then \(\nu \circ d \in \text{Hom}(L^+_\kappa, (2^0_0)^\lambda)\). It must be shown that \(h' := h \circ d\) is in \(Y_\kappa\).

Assume \(h' \not\in Y_\kappa\); then there exist \(x, y \in G^+_\kappa\) such that \(x \rightarrow^+ y\), \(h'(x) = 1\), and \(h'(y) = 0\). Because \(h' \in \text{Hom}(G^+_\kappa, 2^0_0)\) we get \(x, y \in G_\kappa\), \(x \rightarrow y\) and \(x \not\rightarrow y\). Choose \(\{a_i : i < \kappa\} \cup \{b_i : i < \kappa\}\) for \(x, y\) as in the statement of Lemma 4.1(3).

\(L^+_\kappa\) is a chain, and \(\nu \circ d \in \text{Hom}(L^+_\kappa, (2^0_0)^\lambda)\); hence the image of \(\nu \circ d\) in \((2^0_0)^\lambda\) is also a chain. It follows that \(\ker(\nu \circ d) = \ker(d)\) partitions \(L^+_\kappa\) into fewer than \(\kappa\) many intervals. In particular, there must exist \(i < \kappa\) such that \(d(a_i) = d(b_i)\). But \(x \rightarrow^+ a_i\) and \(b_i \rightarrow^+ y\) then imply \(h'(x) \leq h'(a_i) = h'(b_i) \leq h'(y)\), a contradiction. \(\square\)

Fix an infinite cardinal \(\kappa\) and let \(M\) be an alter ego for \(\mathcal{M}\) whose signature consists of relations, operations and partial operations of arities less than \(\kappa\). Let \(G_\kappa\), \(L_\kappa\) and \(Y_\kappa\) be as above, and define \(X_\kappa = \{i \beta_{G^+_\kappa}(y) : y \in Y_\kappa\} \subseteq M^B(G^+_\kappa)\). Note that \(B(G^+_\kappa) \leq B(L^+_\kappa)\) and \(X_\kappa = \{\alpha \mid B(G^+_\kappa) : \alpha \in \text{Hom}(B(L^+_\kappa), M)\}\). These facts automatically imply that \(X_\kappa\) is topologically closed in \(M^B(G^+_\kappa)\). By Lemmas 3.5 and 4.2, \(X_\kappa\) is closed under all algebraic operations of \(M\) of arities less than \(\kappa\). Hence \(X_\kappa\) is a subuniverse of \(M^B(G^+_\kappa)\) and the corresponding substructure \(X_\kappa\) belongs to the topological quasivariety generated by \(M\).

Clearly \(X_\kappa \subseteq \text{Hom}(B(G^+_\kappa), M)\), and the elements of \(X_\kappa\) separate the points of \(B(G^+_\kappa)\); that is, for any \(a, b \in B(G^+_\kappa)\) with \(a \neq b\) there exists \(\alpha \in X_\kappa\) such that \(\alpha(a) \neq \alpha(b)\). It should also be clear that \(X_\kappa \not\in \text{Hom}(B(G^+_\kappa), M)\), since \(Y_\kappa \not\in \text{Hom}(G^+_\kappa, 2^0_0)\).

When \(\kappa = \omega\) these facts plus Lemma 3.8 from [2] imply that \(M\) is not strongly dualizable. To prove that \(M\) is not fully dualizable, indeed is not fully dualized by \(M\), further argument is needed.

**Definition 4.3.** Suppose \(I \neq \emptyset\) and \(X\) is a substructure of \(\langle M, \leq, E, R, \ldots \rangle'\), where \(\leq, E, R\) are as defined in section 2 and \(\langle M, \leq, E, R, \ldots \rangle\) is any alter ego for \(\mathcal{M}\) whose signature includes \(\leq, E, R\). The bi-graph associated with \(X\), denoted \(bg(X)\), is defined as follows. Let \(G_X = \{\alpha \in X : X \models \exists y \forall z ([z \leq \alpha \& \neg(z = \alpha)] \iff z \leq y)\}\).

For \(\alpha \in G_X\) define \(\alpha_*\) to be the unique lower cover of \(\alpha\) in \(\langle X, \leq \rangle\). Define relations \(\rightarrow_X\) and \(\vdash_X\) on \(G_X\) as follows: for \(\alpha, \beta \in G_X\)

\[\alpha \rightarrow_X \beta \iff \beta \leq \alpha\] and

\[\alpha \vdash_X \beta \iff \beta \leq \alpha\]

Then \(bg(X) = \langle G_X, \rightarrow_X, \vdash_X \rangle\).
Lemma 4.4. Suppose \( G = \langle G, \to \rangle \in \mathcal{D} \) and \( G_1 = \langle G, \to \rangle \in \mathcal{D} \) have the same universe and \( \to \) is included in \( \to \). Define
\[
\vdash = \text{the reflexive transitive closure of } \to \text{ in } G
\]
\[
Y = \text{Hom}(G_1^+, 2_{01}) \subseteq \text{Hom}(G^+, 2_{01})
\]
\[
X = \{ h(x) \in Y \subseteq \text{Hom}(B(G^+), M) \}.
\]

If \( X \) is a subuniverse of \( \langle M, \leq, E, R, \ldots \rangle^{B(G^+)} \) and \( X \) is the corresponding substructure, then

1. \( \text{bg}(X) \cong \langle G, \to, \vdash \rangle \).
2. In particular, if \( G_1 = G \) so that \( X = \text{Hom}(B(G^+), M) \), then \( \vdash_X \) is the reflexive transitive closure of \( \to_X \) in \( \text{bg}(X) \).

Proof. Let \( P \) be the poset \( \langle G, \vdash \rangle \) and note that \( Y = \text{Hom}(P^+, 2_{01}) \). For each \( a \in G \) define \( h_a, h'_a \in Y \) by
\[
h_a(x) = \begin{cases} 
1 & \text{if } a \vdash x \\
0 & \text{otherwise}
\end{cases}
\]
\[
h'_a(x) = \begin{cases} 
1 & \text{if } a \vdash x \text{ and } x \neq a \\
0 & \text{otherwise}
\end{cases}
\]

and let \( J = \{ h_a : a \in G \} \). As is well-known from the theory of posets,
\[
J = \text{the set of completely join-irreducible elements of } \langle Y, \leq \rangle
\]
\[
a \vdash b \iff h_b \leq h_a, \quad \text{for } a, b \in G
\]
\[
h'_a = \text{the unique lower cover of } h_a \text{ in } \langle Y, \leq \rangle, \quad \text{for } a \in G.
\]

On the other hand, note that \( \langle Y, \leq \rangle \) is a subposet of \( \langle \text{Hom}(G^+, 2_{01}), \leq \rangle \). Thus by Lemma 3.1(3), the map \( h \mapsto \iota \beta_{G^+}(h) \) is an order isomorphism \( \varphi : \langle Y, \leq \rangle \to \langle X, \leq \rangle \) and so
\[
G_X = \{ \varphi(h_a) : a \in G \}
\]
\[
a \vdash b \iff \varphi(h_b) \vdash_X \varphi(h_a)
\]
\[
\varphi(h'_a) = \text{the unique lower cover of } \varphi(h_a) \text{ in } X.
\]

Our isomorphism \( \langle G, \to, \vdash \rangle \cong \text{bg}(X) \) will be the map \( a \mapsto \varphi(h_a) \). It remains to prove that if \( a, b \in G \) with \( a \vdash b \) and \( a \neq b \), then
\[
a \to a \iff X \models \neg E(\varphi(h'_a), \varphi(h_a)), \quad \text{while}
\]
\[
a \to b \iff X \models \neg R(\varphi(h'_a), \varphi(h'), \varphi(h'), \varphi(h_a)).
\]

We shall prove the second equivalence, the first being similar. Let \( \beta_s, \beta, \alpha_s, \alpha \) be the 4-tuple \( \langle \varphi(h'_a), \varphi(h_b), \varphi(h'_a), \varphi(h_a) \rangle \). Since \( a \vdash b \) and \( a \neq b \) we have \( h'_b \leq h_b \leq h'_a \leq h_a \) and thus \( \beta_s \leq \beta \leq \alpha_s \leq \alpha \). Thus the only way that \( R(\beta_s, \beta, \alpha_s, \alpha) \) can fail to be true is if at some coordinate \( (x, y) \in B(G^+) \) we have \( \beta_s(x, y) = 0, \beta(x, y) = \alpha_s(x, y) = 1, \)
and \( \alpha(x, y) = 2 \). Note that for all \((x, y) \in B(G^+)\) we have \( x \rightarrow y \) and therefore \( x \vdash y \); thus for \( c \in \{a, b\} \) we have
\[
\varphi(h_c)(x, y) = \begin{cases} 
2 & \text{if } c \vdash x \\
1 & \text{if } c \vdash y \text{ and } c \nvdash x \\
0 & \text{if } c \nvdash y
\end{cases}
\]
\[
\varphi(h'_c)(x, y) = \begin{cases} 
2 & \text{if } c \vdash x \text{ and } c \neq x \\
1 & \text{if } c \vdash y \text{ and } c \neq y \text{ and either } c \nvdash x \text{ or } c = x \\
0 & \text{if } c \nvdash y \text{ or } c = y
\end{cases}
\]

It follows that \( (\beta_*(x, y), \beta(x, y), \alpha_*(x, y), \alpha(x, y)) = (0, 1, 2) \) iff \((x, y) = (a, b)\); so \( X \models \neg R(\beta_*, \beta, \alpha_*, \alpha) \) iff \((a, b) \in B(G^+)\), which is equivalent to \( a \rightarrow b \).

\section*{Definition 4.5}
Suppose \( A \) is a finite algebra, \( S \leq A^n \), and \( S \) is the corresponding \( n \)-ary relation on \( A \). \( S \) is balanced if \( |\text{Hom}(S, A)| = n \) and \( S \) has no repeated coordinates; i.e., \( \rho_i|_S \neq \rho_j|_S \) whenever \( i \neq j \), where \( \rho_1, \ldots, \rho_n \) are the projections \( A^n \rightarrow A \).

Note that \( \text{Hom}(S, A) = \{\rho_i|_S : 1 \leq i \leq n\} \) if \( S \) is balanced.

\section*{Lemma 4.6}
\( \leq, E, \) and \( R \) are balanced for \( M \).

\textit{Proof.} Here is a way to compute the sizes of the relevant hom-sets “by inspection.” Let \( L \) denote the subalgebra of \( M^2 \) whose universe is \( \leq \). Then \( L, E, R \) are isomorphic to \( B(G_1^+), B(G_2^+), B(G_3^+) \) respectively, where \( G_1, G_2, G_3 \) are pictured below:

\begin{center}
\begin{tikzpicture}
\node (G1) at (0,0) {G_1};
\node (G2) at (2,0) {G_2};
\node (G3) at (4,0) {G_3};
\end{tikzpicture}
\end{center}

Now use Lemma 3.1(3).

\section*{Lemma 4.7}
Suppose \( A \) is a finite algebra, \( \mathbb{A} \) is an alter ego (whose relations and operations may be infinitary) which dualizes \( A \), and \( \mathbb{A}^* \) is an alter ego obtained from \( \mathbb{A} \) by adding one or more balanced relations of \( \mathbb{A} \) to the signature.

1. If \( S \) is a balanced \( n \)-ary relation of \( A \), then \( S \) is defined in \( \mathbb{A} \) by a finite conjunction \( \Phi(x_1, \ldots, x_n) \) of atomic formulas in the signature of \( \mathbb{A} \).

2. \( \mathbb{A} \) fully dualizes \( A \) iff \( \mathbb{A}^* \) fully dualizes \( A \).

\textit{Proof.} (1). The argument is essentially due to Zádori [6, Corollary 3.2 and Theorem 3.3] and, independently, Davey, Haviar and Priestley [4, Theorem 3.6]. Write \( \mathbb{A} = \langle A, \mathcal{R}, \mathcal{F}, \mathcal{C}, \text{topology} \rangle \) where \( \mathcal{R}, \mathcal{F}, \mathcal{C} \) are the sets of relations, functions of positive
arity, and constants respectively that are included in the signature of A. Define
\[ X = \text{Hom}(S, A) \subseteq A^S \]
\[ S^* = \{ e^S(a) : a \in S \} \subseteq A^X. \]

Since A dualizes A,
\[ S^* = \{ \varphi \in A^X : \varphi \text{ preserves } R \cup F \cup C \}. \]

Now for \( x = (x_1, \ldots, x_n) \in A^n \) define \( \varphi_x \in A^X \) by \( \varphi_x(\rho_i|_S) = x_i. \) Note that the map \( x \mapsto \varphi_x \) is well-defined (as \( S \) is balanced) and injective, and that \( \varphi_a = e^S(a) \) for each \( a \in S. \) Thus
\[ S = \{ x \in A^n : \varphi_x \in S^* \} \]
\[ = \{ x \in A^n : \varphi_x \text{ preserves } R \cup F \cup C \}. \]

This last equation will provide the desired first-order formula. To see how, suppose \( c \in C. \) Then \( \{ c \} \) is a one-element subalgebra of A. Hence among the homomorphisms from \( S \) to \( A \) is one which is the constant map with range \( \{ c \}. \) Choose \( i \) so that \( \rho_i|_S \) is this homomorphism (equivalently, so that \( a_i = c \) for all \( a \in S \)). Then for any \( x \in A^n, \) \( \varphi_x \) preserves \( c \) iff \( x_i = c. \) Define \( \Phi_c(x_1, \ldots, x_n) \) to be the atomic formula \( x_i = c. \) Then \( \varphi_x \) preserves \( c \) iff \( A \models \Phi_c(x). \)

We give a similar argument for each member of \( R \cup F. \) If \( R \) is a \( \lambda \)-ary relation in \( R, \) define
\[ \Omega_R = \{ \sigma \in \{1, \ldots, n\}^\lambda : (a_{\sigma(i)})_{i<\lambda} \in R \text{ for all } a \in S \}. \]

Then define \( \Phi_R(x_1, \ldots, x_n) \) to be \( \bigwedge \{ R((x_{\sigma(i)})_{i<\lambda}) : \sigma \in \Omega_R \}. \) As before, \( \varphi_x \) preserves \( R \) iff \( A \models \Phi_R(x). \) Finally, if \( F \) is a \( k \)-ary operation in \( F, \) define
\[ \Omega_F = \{ (\sigma, j) \in \{1, \ldots, n\}^{\lambda+1} : F((a_{\sigma(i)})_{i<\lambda}) = a_j \text{ for all } a \in S \}. \]

Then define \( \Phi_F(x_1, \ldots, x_n) \) to be \( \bigwedge \{ F((x_{\sigma(i)})_{i<\lambda}) = x_j : (\sigma, j) \in \Omega_F \}. \) As before, \( \varphi_x \) preserves \( F \) iff \( A \models \Phi_F(x). \) Thus a suitable formula which defines \( S \) in \( A \) is
\[ \bigwedge_{R \in R} \Phi_R(x_1, \ldots, x_n) \& \bigwedge_{F \in F} \Phi_F(x_1, \ldots, x_n) \& \bigwedge_{c \in C} \Phi_c(x_1, \ldots, x_n). \]

Since \( S \subseteq A^n \) and \( A^n \) is finite, only finitely many of the atomic conjuncts in the above formula are needed to define \( S \) in \( A. \)

(2). Clearly \( A^* \) continues to dualize \( A \) (see [1, Lemma 2.4.2 and Theorem 2.4.3(i)]), and \( A^I \) and \( (A^*)^I \) have the same topologically closed subuniverses for any set \( I. \) What remains to be shown is the following: if \( X, Y \) are topologically closed subuniverses of \( A^I, A^J \) respectively, \( X, Y \) are the corresponding topological substructures, and \( X^*, Y^* \) are the corresponding topological substructures of \( (A^*)^I, (A^*)^J \) respectively, then \( X \cong Y \) iff \( X^* \cong Y^*. \) That this is true follows from item 1. \( \square \)

We now have all the ingredients needed to complete our argument.
Theorem 4.8. M is dualizable but is not fully dualized by any alter ego having only a set of relations, operations and partial operations in its signature.

Proof. Suppose M is fully dualized by an alter ego \( \mathbb{M} \) whose signature is a set. By Lemma 4.7(2), we may assume that \( \leq, E \) and \( R \) are included in the signature of \( \mathbb{M} \). Let \( \kappa \) be an infinite cardinal greater than all the arities of the relations, operations and partial operations in this signature. Let \( \mathbb{G}_\kappa = \langle G_\kappa, \rightarrow, \rightarrow' \rangle \) be as in Lemma 4.1, and let \( \mathbb{X}_\kappa \) be as in the discussion preceding Definition 4.3. Since \( \mathbb{M} \) fully dualizes \( M \) there exists \( A \in \text{ISP}(M) \) so that, with \( X' \) denoting \( \text{Hom}(A, M) \) as a topological substructure of \( (M)^A \), we have \( \mathbb{X}_\kappa \cong X' \). It follows that \( \text{bg}(\mathbb{X}_\kappa) \cong \text{bg}(X') \). Using Lemma 4.4 twice and Lemma 3.2(3) we find that \( X' \) is the reflexive transitive closure of \( \rightarrow \) in \( \text{bg}(X') \), while \( \text{bg}(\mathbb{X}_\kappa) \cong G_\kappa \). Since \( \rightarrow' \) is not the reflexive transitive closure of \( \rightarrow \) in \( G_\kappa \), we have our desired contradiction. \( \square \)

References