ON FINITELY BASED GROUPS AND NONFINITELY BASED QUASIVARIETIES

JOHN LAWRENCE AND ROSS WILLARD

ABSTRACT. We define the notion of *strictly finitely based* varieties of groups, and determine which finite groups are strictly finitely based. Then we use this result to prove the existence of 'very nonfinitely based' finite algebras, thereby solving two open problems in the theory of quasivarieties.

1. INTRODUCTION

A *variety* of groups is any class of groups which can be defined by some collection of *identities* (universally quantified equations $w \approx 1$); such a defining collection is a *basis* for the variety. From the point of view of equational logic, if Σ is a basis for the variety $\mathcal{V}$ then the identities true in $\mathcal{V}$ are precisely the formally derivable consequences of Σ. A key feature of formal derivations is that the members $w(x_1, \ldots, x_k) \approx 1$ of Σ are typically replaced by arbitrary substitution instances $w(w_1, \ldots, w_k) \approx 1$ (where $w_1, \ldots, w_k$ are words); this corresponds to the formation of a verbal subgroup in a free group of suitable rank (see [11]).

A natural question to ask of a variety $\mathcal{V}$ is whether it is *finitely based*, i.e., has a finite basis. In this paper we ask a related question: whether $\mathcal{V}$ has a finite basis from which all identities of $\mathcal{V}$ can be proved (modulo the group axioms) via string-rewrite derivations in which substitutions are restricted to words w_i of some bounded length. We call a variety of groups with such a basis *strictly finitely based.* In Section 2 we determine which finite groups G have the property that $\mathcal{V}(G)$ (the variety defined by the identities true in G) is strictly finitely based.

The rest of our paper is devoted to solving two open problems in the theory of quasivarieties. The first problem appeared in 1981 in a paper of W. Rautenberg [14, Problem 5] and is attributed to A. Wroński in [13, Problem 9.9].

QUESTION (Rautenberg, Wroński): *Does there exist a finite algebra* $\mathbf{A}$ *for which there is no finitely based quasivariety* $\mathcal{Q}$ *satisfying* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathcal{Q})$*?*

Definitions of all terms appearing here may be found in [3]. Briefly, 'algebra' is meant in the general sense of universal algebra or model theory, with the added

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proviso that our algebras have only finitely many primitive operations. A *quasivariety* is any class of such algebras which can be defined by some collection of universally quantified implications between equations (or between conjunctions of equations); the class of torsion-free groups is an example. A quasivariety is *finitely based* if the set of defining implications can be chosen to be finite. $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathcal{Q})$ mean the varieties defined by the identities true in $\mathbf{A}$ and the identities true in all members of $\mathcal{Q}$ respectively.

The Rautenberg-Wroński question arose naturally in the study of algebraic deductive systems in logic. In the same context, D. Pigozzi asked [13, p. 532] the following related question, which is discussed at greater length in [8]:

Question (Pigozzi): *Does there exist a finite algebra which does not belong to any finitely based, locally finite quasivariety?*

In Sections 3 and 4 of this paper we apply the results of Section 2 and a technique from [6] to construct an 18-element algebra $\mathbf{A}^*$ (a semigroup with one extra unary operation) having the property that if $\mathcal{K}$ is any class of algebras of the same type satisfying $\mathcal{V}(\mathbf{A}^*) = \mathcal{V}(\mathcal{K})$, then $\mathcal{K}$ is not contained in any finitely based, locally finite quasivariety. This answers the Rautenberg-Wroński and Pigozzi questions affirmatively.

2. Strictly finitely based groups

Let $\mathcal{V}$ be a variety of groups and W a set of words w identically equal to 1 in $\mathcal{V}$ (i.e., such that $w \approx 1$ is an identity of $\mathcal{V}$). We say that W is a *strict basis* for $\mathcal{V}$ if, modulo the associative law, every identity of $\mathcal{V}$ can be deduced from $\{w \approx 1 : w \in W\}$ via derivations in which substitutions are restricted to replacements of variables by variables.

More precisely, fix a sequence of variables $x_1, x_2, \dots$ and assume that each $w \in W$ is a word in these variables. For each $m > 0$ define $P(W, m)$ to be the group presented by the generators $x_1, \dots, x_m$ and all relations $w(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ where $w(x_1, \dots, x_k) \in W$ and $\sigma : \{1, \dots, k\} \to \{1, \dots, m\}$. Then W is a strict basis for $\mathcal{V}$ if and only if (i) each $w \in W$ is identically equal to 1 in $\mathcal{V}$, and (ii) either of the following equivalent conditions holds:

(1) For every $m > 0$, $P(W, m)$ is the relatively free group $\mathbb{F}_{\mathcal{V}}(m)$ in $\mathcal{V}$ of rank m.
(2) For any group G and set S such that $G = \langle S \rangle$, if $G \models w(a_1, \dots, a_n) = 1$ whenever $w(x_1, \dots, x_n) \in W$ and $a_1, \dots, a_n \in S$, then $G \in \mathcal{V}$.

We say that $\mathcal{V}$ is *strictly finitely based* if there exists a finite set of words which is a strict basis for $\mathcal{V}$. A group is strictly finitely based if the variety it generates is.[1]

[1]This should not be confused with Rautenberg's notion of a *strongly finitely based* equational theory [15].

Our first result strengthens [11, Corollary 34.13]. Recall that the *left-normed higher commutators* are defined by $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ for $n > 2$.

Lemma 2.1. *Suppose $\mathcal{V}$ is a variety of nilpotent groups of class $\leq c$ ($c > 0$). Let W be the set consisting of $[x_1, \ldots, x_{c+1}]$ and all words in the variables $x_1, \ldots, x_c$ identically equal to 1 in $\mathcal{V}$. Then W is a strict basis for $\mathcal{V}$.*

Proof. For any group G and a subset $S \subseteq G$, define $S_{[1]} = S$ and $S_{[n+1]} = \{[g, s] : g \in S_{[n]},\ s \in S\}$ for $n \geq 1$. Our notation for the nth member of the lower central series for G is $G_{(n)}$ with $G_{(1)} = G$.

Claim 1. If $G = \langle S \rangle$ and $S_{[c+1]} = (1)$, then G is nilpotent of class $\leq c$.
Claim 2. If $G = \langle S \rangle$ and G is nilpotent of class $\leq c$, then $G_{(c)} = \langle S_{[c]} \rangle$.

Indeed, if $S_{[c+1]} = (1)$ then every member of $S_{[c]}$ is central; thus by induction on c, $G/C(G)$ is nilpotent of class $\leq c - 1$, which proves Claim 1. Claim 2 can be found in e.g. [18, p. 42].

Now we prove the Lemma by induction on c. Fix $m > 0$; let $P = P(W, m)$ and $\mathbb{F} = \mathbb{F}_{\mathcal{V}}(m)$. Clearly $P = \mathbb{F}$ if $m \leq c$, so assume $m > c$. Let U be the set of all commutator words $[x_{\sigma(1)}, \ldots, x_{\sigma(c)}]$ where $\sigma : \{1, \ldots, c\} \to \{1, \ldots, m\}$, unless $c = 1$ in which case let $U = \{x_1, \ldots, x_m\}$. For any word w in $x_1, \ldots, x_m$ let $\tilde{w}$ and $\hat{w}$ denote the images of w in P and $\mathbb{F}$ respectively, and let $\alpha : P \to \mathbb{F}$ be the map $\tilde{w} \mapsto \hat{w}$. To prove the Lemma it suffices to show that α is injective.

Let $\mu : P \to P/P_{(c)}$ and $\nu : \mathbb{F} \to \mathbb{F}/\mathbb{F}_{(c)}$ be the canonical maps. Then $\beta : P/P_{(c)} \to \mathbb{F}/\mathbb{F}_{(c)}$ satisfying $\beta\mu = \nu\alpha$ is well-defined. We first claim that β is injective. This is obvious if $c = 1$. If $c > 1$, let $\mathcal{N}_{c-1}$ be the variety of all nilpotent groups of class $\leq c - 1$ and let $\mathcal{V}' = \mathcal{V} \cap \mathcal{N}_{c-1}$. Then $\mathbb{F}_{\mathcal{V}'}(m) = \mathbb{F}/\mathbb{F}_{(c)}$. By induction, the set W' of all words in $x_1, \ldots, x_c$ identically equal to 1 in $\mathcal{V}'$ is a strict basis for $\mathcal{V}'$.

Suppose $w \in W'$. Then $\hat{w} \in \mathbb{F}_{(c)}$. Thus by Claim 2, $\hat{w} = \hat{u}_1^{\epsilon_1}\hat{u}_2^{\epsilon_2}\cdots\hat{u}_k^{\epsilon_k}$ for some $u_j \in U$ and $\epsilon_j \in \{1, -1\}$. Then $w \approx u_1^{\epsilon_1}u_2^{\epsilon_2}\cdots u_k^{\epsilon_k}$ is an identity of $\mathcal{V}$, and by replacing $x_{c+1}, \ldots, x_m$ by 1 we can assume it is an identity in the variables $x_1, \ldots, x_c$ only. Hence $w^{-1}u_1^{\epsilon_1}\cdots u_k^{\epsilon_k} \in W$, so $w(\tilde{x}_{\sigma(1)}, \ldots, \tilde{x}_{\sigma(c)}) \in P_{(c)}$ whenever $\sigma : \{1, \ldots, c\} \to \{1, \ldots, m\}$. These remarks prove $P/P_{(c)} \in \mathcal{V}'$ and hence β is injective.

Now we turn to showing that α is injective. Let $S = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$; then $P = \langle S \rangle$, and $S_{[c+1]} = (1)$ since $[x_1, \ldots, x_{c+1}] \in W$.

Claim 3. Suppose w_1, w_2 are words in $x_1, \ldots, x_m$ such that w_2 is a substitution instance of w_1. If $\tilde{w}_1 \in \ker\alpha$ then $\tilde{w}_2 \in \ker\alpha$.
Claim 4. $\ker\alpha \subseteq P_{(c)}$.
Claim 5. $P_{(c)} = \langle S_{[c]} \rangle = \langle \{\tilde{u} : u \in U\} \rangle$.

Indeed, Claim 3 is true because the image of α is a relatively free group, Claim 4 follows from the fact that β is injective, and Claim 5 is a consequence of Claims 1 and 2.

Now suppose $\ker\alpha \neq (1)$. By Claims 4 and 5, there exist $u_0, \ldots, u_k \in U$ and $\epsilon_i \in \{1, -1\}$ such that $1 \neq \tilde{u}_0^{\epsilon_0} \cdots \tilde{u}_k^{\epsilon_k} \in \ker\alpha$. Let the u_j, ϵ_j be so chosen with k as small as possible, and let $w_1 = u_0^{\epsilon_0} \cdots u_k^{\epsilon_k}$. Let the distinct variables occurring in u_0 be $x_{i_1}, \ldots, x_{i_p}$ $(p \leq c)$. To simplify matters we assume that $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$. Because the $\tilde{u}_j$ are central in P, we can assume that there exists a nonnegative integer $r \leq k$ such that for all $j \leq k$, the variables in u_j are contained in $\{x_1, \ldots, x_p\}$ if and only if $j \leq r$. Let w_2 be the word obtained from w_1 by replacing $x_{p+1}, \ldots, x_m$ by 1. Then $\tilde{w}_2 = \tilde{u}_0^{\epsilon_0} \cdots \tilde{u}_r^{\epsilon_r}$ and $\tilde{w}_2 \in \ker\alpha$ by Claim 3, hence also $\tilde{u}_{r+1}^{\epsilon_{r+1}} \cdots \tilde{u}_k^{\epsilon_k} \in \ker\alpha$. If $j < r$, then $\tilde{u}_0^{\epsilon_0} \cdots \tilde{u}_r^{\epsilon_r} = \tilde{u}_{r+1}^{\epsilon_{r+1}} \cdots \tilde{u}_k^{\epsilon_k} = 1$ by the minimality of k, which contradicts $\tilde{w}_1 \neq 1$. So $j = r$. Thus w_1 is a word in $x_1, \ldots, x_p$ only. $\tilde{w}_1 \in \ker\alpha$ implies that $w_1 \approx 1$ is an identity of $\mathcal{V}$, hence $w_1 \in W$, which contradicts $\tilde{w}_1 \neq 1$. Thus α is injective. □

Now we turn to finite groups. Given a finite group G and integers $m > k > 0$ define $P_G(k, m) = P(W, m)$ where W is the set of all words in the variables $x_1, \ldots, x_k$ identically equal to 1 in G. As $\mathcal{V}(G)$ is locally finite, G is strictly finitely based if and only if there exists an $n \in \mathbb{N}$ such that for all $m > n$, $P_G(n, m) = \mathbb{F}_{\mathcal{V}(G)}(m)$. Thus:

Corollary 2.2. *Every finite nilpotent group is strictly finitely based.*

By the theorem of S. Oates and M. B. Powell [12], every finite group is finitely based. However, not every finite group is strictly finitely based. One way a finite group G might fail to be strictly finitely based is if, for every $n > 0$, there exists $m > n$ such that $P_G(n, m)$ is infinite. Let us call a finite group with this property *inherently not strictly finitely based.*

Recall [11] that $\mathcal{A}_n\mathcal{A}_m$ is the variety of all extensions of abelian groups of exponent dividing n by abelian groups of exponent dividing m.

Lemma 2.3. *Suppose G is a finite group such that $\mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{V}(G)$ for some primes p, q. Then G is inherently not strictly finitely based.*

REMARK: Primes p, q satisfying the hypothesis of Lemma 2.3 must be distinct (see [11, 24.63] or [7]).

Proof. We shall construct, for each $n \geq 2$, an infinite group G_n^* generated by an $(n+1)$-element set X, such that for every n-element subset $Y \subset X$ the subgroup of G_n^* generated by Y is in $\mathcal{V}(G)$.

Suppose $\langle V, E\rangle$ is a finite graph, i.e, V is a finite nonempty set and E is a set of 2-element subsets of V. $Gr(V, E)$ shall denote the group presented by the set of generators V and the set of relations

$$\{v^p : v \in V\} \cup \{[v,w] : \{v,w\} \in E\}.$$

Since each generator has finite order, every element of $Gr(V,E)$ is a product of generators. Observe that:

(1) Distinct elements of V remain distinct in $Gr(V,E)$.
(2) If E is not the set of all 2-element subsets of V, then $Gr(V,E)$ contains the free product $C_p * C_p$ as a retract, and hence is infinite. (C_p is the cyclic group of order p.)
(3) Every automorphism of $\langle V,E\rangle$ extends to an automorphism of $Gr(V,E)$.

Let $n \geq 2$ be fixed. Define V to be the direct product

$$V = C_q \times C_q \times \cdots \times C_q \quad (n \text{ factors})$$

with standard generating set $\{x_1,\ldots,x_n\}$ and identity element 1. For $1 \leq i \leq n$ let V_i be the subgroup of V generated by $\{x_1,\ldots,x_n\} \setminus \{x_i\}$. Then define

$$E = \{\{w_1,w_2\} : w_1 \neq w_2 \text{ and } w_1w_2^{-1} \in V_i \text{ for some } i\}$$

and $G_n = Gr(V,E)$. By an earlier observation, G_n is infinite.

Convention: To distinguish the product of two elements in V from their product in G_n, we denote the set of generators of G_n by $[V] = \{[w] : w \in V\}$.

For each $w \in V$ define $\phi_w : [V] \to [V]$ by $\phi_w([w']) = [ww']$. Then ϕ_w is an automorphism of the graph $\langle [V],E\rangle$, and hence extends to an automorphism ϕ^*_w of G_n. Moreover, the map $\phi^* : w \mapsto \phi^*_w$ is a group homomorphism from V to $\operatorname{Aut} G_n$. Define G^*_n to be the semidirect product of G_n by V with respect to ϕ^*. Thus (i) every element of G^*_n is uniquely expressible in the form wg where $w \in V$ and $g = [w_1][w_2]\cdots[w_k] \in G_n$; (ii) $[w_1]w = w[ww_1]$ for $w,w_1 \in V$; (iii) G^*_n is infinite.

Clearly G^*_n is generated by the $(n+1)$-element set $X = \{x_1,\ldots,x_n,[1]\}$. We finish the proof by showing that if Y is any n-element subset of X then the subgroup generated by Y is in $\mathcal{V}(G)$. If $Y = \{x_1,\ldots,x_n\}$ then

$$\langle Y\rangle = V = (C_q)^n \in \mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{V}(G).$$

On the other hand, if $Y = X \setminus \{x_i\}$ then

$$\langle Y\rangle = \{w[w_1][w_2]\cdots[w_k] : w,w_1,\ldots,w_k \in V_i\}.$$

By design, the subgroup of G_n generated by $\{[w] : w \in V_i\}$ is an abelian group of exponent p. $\langle Y\rangle$ is an extension of this subgroup by $V_i \cong (C_q)^{n-1}$, so $\langle Y\rangle \in \mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{V}(G)$ in this case as well. □

Theorem 2.4. *Let G be a finite group. The following conditions are equivalent:*

(1) *G is not nilpotent;*

(2) *G is not strictly finitely based;*
(3) *G is inherently not strictly finitely based;*
(4) *$\mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{V}(G)$ for some distinct primes p, q.*

Proof. (4) $\Rightarrow$ (3) is Lemma 2.3, (3) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (1) is Corollary 2.2. To prove (1) $\Rightarrow$ (4), assume G is finite and not nilpotent. Choose a nonnilpotent $G_0 \in \mathcal{V}(G)$ of minimum cardinality. We claim that $\mathcal{V}(G_0) = \mathcal{A}_p\mathcal{A}_q$ for some distinct primes p, q. Begin by following the argument in [2, Lemma 3.6]. As the proper subgroups of G_0 are nilpotent, a theorem in [16, p. 148] says that G_0 is solvable. Thus we can pick a normal subgroup H of G_0 such that G_0/H is cyclic of order q, where q is a prime. Choose $g \in G_0$ so that gH is a generator of G_0/H. By taking an appropriate power of g if necessary, we can assume that the order of g is q^m for some m. Since G_0 is not nilpotent, there exists a prime $p \neq q$ dividing the order of H. Let S be the unique Sylow p-subgroup of H, and let C be the center of S. Then C is contained in the center of H (since H is nilpotent), is normal in G_0 (since C is a characteristic subgroup of H), and is nontrivial, say of order p^k, $k \geq 1$.

If $cg = gc$ for all $c \in C$ then C would be contained in the center of G_0. As G_0/C is nilpotent, this would imply G_0 is nilpotent, which is false. So $gc \neq cg$ for some $c \in C$. Now let $G_1 = \langle C \cup \{g\}\rangle$. Then G_1 has order $p^k q^m$, C is its unique Sylow p-subgroup, and $\langle g\rangle$ is a Sylow q-subgroup. G_1 is not nilpotent and thus $G_1 = G_0$, which proves that all Sylow subgroups of G_0 are abelian. This property is inherited by every subgroup of G_0; hence every proper subgroup of G_0, being nilpotent, is abelian. By the results of [17] (arguing as in [7] or [4, Propos. 1]) it follows that $\mathcal{V}(G_0) = \mathcal{A}_p\mathcal{A}_q$. □

3. 2-sorted algebras

By a 2-*sorted algebra of type* τ we mean any structure $\mathbf{A} = \langle A^{(0)}, A^{(1)}, b\rangle$ where $A^{(0)}$ and $A^{(1)}$ are nonempty sets and b is a function $A^{(1)} \times A^{(0)} \to A^{(0)}$. For $x \in A^{(1)}$ and $y \in A^{(0)}$ we may write $x{\cdot}y$ in place of $b(x, y)$.

We remind the reader of the basic facts about such algebras (see e.g. [1] or [19]). A *homomorphism* from $\mathbf{A}$ to a similar algebra $\mathbf{B}$ is a pair of maps $f^{(i)} : A^{(i)} \to B^{(i)}$ satisfying $f^{(0)}(x{\cdot}y) = f^{(1)}(x){\cdot}f^{(0)}(y)$ for all $x \in A^{(1)}$ and $y \in A^{(0)}$. Direct products of families of 2-sorted algebras of type τ are defined in the obvious way. If $\mathbf{A}$ is an algebra of type τ and $U^{(i)}$ is a nonempty subset of $A^{(i)}$ for $i = 0, 1$, then the *subalgebra of* $\mathbf{A}$ *generated by* $\langle U^{(0)}, U^{(1)}\rangle$ is the algebra $\langle C, U^{(1)}, b|_{U^{(1)}\times C}\rangle$ where C is the closure of $U^{(0)}$ under the action of $U^{(1)}$. If κ_0, κ_1 are nonzero cardinals, then we say that $\mathbf{A}$ is (κ_0, κ_1)-*generated* if there exist $U^{(i)} \subseteq A^{(i)}$ with $|U^{(i)}| \leq \kappa_i$ such that $\langle U^{(0)}, U^{(1)}\rangle$ generates $\mathbf{A}$; and is *finitely generated* if it is (n_0, n_1)-generated for some $n_i < \omega$.

The sets of 2-sorted identities and quasi-identities of type τ are defined in the obvious way, using distinct variables for each of the two sorts. Then the usual development of varieties and quasivarieties of algebras of type τ proceeds as in the 1-sorted case. In particular, if $\mathcal{V}$ is a variety of type τ, then for each pair of nonzero cardinals κ, λ the (κ, λ)-generated $\mathcal{V}$-free algebra $\mathbb{F}_{\mathcal{V}}(\kappa, \lambda)$ exists (in $\mathcal{V}$) and is constructed in the standard way.

There is a useful correspondence between 2-sorted algebras of type τ and certain 1-sorted (i.e., ordinary) algebras, which we now make precise. Let L be the 1-sorted language consisting of one binary operation symbol d and one unary operation symbol f. Given the 2-sorted $\mathbf{A} = \langle A^{(0)}, A^{(1)}, b \rangle$ of type τ, define $\mathbf{A}^*$ to be the L-algebra whose universe is $A^{(0)} \times A^{(1)}$ and whose fundamental operations are given by

$$\begin{aligned} d^{\mathbf{A}^*}(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) &= \langle x_0, y_1 \rangle \\ f^{\mathbf{A}^*}(\langle x_0, x_1 \rangle) &= \langle x_1 \cdot x_0, x_1 \rangle. \end{aligned}$$

$\mathbf{A}^*$ satisfies the following identities:

$$\Sigma \ : \ \begin{cases} d(x,x) \approx x \\ d(d(x,y), d(z,w)) \approx d(x,w) \\ d(x, f(y)) \approx d(x,y). \end{cases}$$

Conversely, every L-algebra satisfying the identities Σ is isomorphic to $\mathbf{A}^*$ for some 2-sorted $\mathbf{A}$ of type τ. In fact, the map $\mathbf{A} \mapsto \mathbf{A}^*$ determines a category equivalence between the category of all 2-sorted algebras of type τ and the variety $\mathcal{V}_\tau$ of 1-sorted L-algebras axiomatized by Σ. Thus $\mathcal{V}(\mathbf{A}^*)$ is the subvariety of $\mathcal{V}_\tau$ consisting of all algebras isomorphic to some member of $\{\mathbf{B}^* : \mathbf{B} \in \mathcal{V}(\mathbf{A})\}$. The map $\mathbf{A} \mapsto \mathbf{A}^*$ also has the property that $\mathbf{A}$ is (n,n)-generated if and only if $\mathbf{A}^*$ is n-generated; hence $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n,n)^* = \mathbb{F}_{\mathcal{V}(\mathbf{A}^*)}(n)$ for all $n > 0$. Finally, easy syntactic translations show that $\mathcal{V}(\mathbf{A}^*)$ is finitely based if and only if $\mathcal{V}(\mathbf{A})$ is.

Remarks:

(1) If $\mathbf{B} = \langle B, d^{\mathbf{B}}, f^{\mathbf{B}} \rangle \in \mathcal{V}_\tau$ then $\langle B, d^{\mathbf{B}} \rangle$ is a semigroup (in fact, a rectangular band).
(2) If $\mathbf{B} \in \mathcal{V}_\tau$ then $d^{\mathbf{B}}$ is called a *binary decomposition operation* on the set B [9, p. 162]. See [10, ch. 11], where the construction of a 2-sorted variety associated with $\mathcal{V}_\tau$ and d is described. In fact, the variety $\mathcal{V}_\tau[d]$ constructed there is term-equivalent to the variety of all 2-sorted algebras of type τ. See [10] for more details concerning the correspondence between $\mathcal{V}_\tau$ and $\mathcal{V}_\tau[d]$.
(3) In the language of tame congruence theory [5], every 2-sorted $\mathbf{A}$ of type τ is 2-*step strongly solvable* (the congruence which collapses $A^{(0)}$ is both

strongly abelian and co-strongly abelian); hence each member of $\mathcal{V}_\tau$ is also 2-step strongly solvable.

4. The construction

Throughout this section G is a fixed finite nontrivial group and $b : G\times S \to S$ is a faithful left action of G on the finite set S. Let $\mathbf{A} = \langle S, G, b\rangle$ be the corresponding 2-sorted algebra of type τ. The free algebras of finite rank in $\mathcal{V}(\mathbf{A})$ can be described very easily. Suppose r, n are positive integers. Let $\mathbb{F}_n = \mathbb{F}_{\mathcal{V}(G)}(n)$ be the relatively free group of rank n in $\mathcal{V}(G)$, with free generators $\{x_1, \ldots, x_n\}$. Let $S_{r,n}$ be a set consisting of r disjoint copies of $\mathbb{F}_n$ and let $b_{r,n}$ be the action of $\mathbb{F}_n$ on $S_{r,n}$ given by left multiplication within each copy. Then $\langle S_{r,n}, \mathbb{F}_n, b_{r,n}\rangle$ is a 2-sorted algebra of type τ.

Lemma 4.1. *$\mathbb{F}_{\mathcal{V}(\mathbf{A})}(r, n)$ is the subalgebra of $\langle S_{r,n}, \mathbb{F}_n, b_{r,n}\rangle$ having universes $S_{r,n}$ and $\{x_1, \ldots, x_n\}$.*

Proof. Let $v_1, \ldots, v_r$ be variables of sort 0, and $x_1, \ldots, x_n$ variables of sort 1. The set $T^{(1)}$ of terms of sort 1 (of type τ, in these variables) is just $T^{(1)} = \{x_1, \ldots, x_n\}$. The set $T^{(0)}$ of terms of sort 0 (in these variables) is more complicated; it consists of (i) the variables $v_1, \ldots, v_r$, and (ii) all expressions of the form $u_1\cdot(u_2\cdot(\cdots\ \cdot(u_t\cdot v_i)\cdots))$ where t is positive, $1 \le i \le r$, and $u_1, \ldots, u_t \in T^{(1)}$. Let $(T^{(1)})^*$ denote the free monoid on $T^{(1)}$. For any string $w = u_1u_2\cdots u_t$ in $(T^{(1)})^*$ and for $1 \le i \le r$, let $[w, i] = v_i$ if $t = 0$ and $[w, i] = u_1\cdot(u_2\cdot(\cdots\ \cdot(u_t\cdot v_i)\cdots))$ otherwise. Thus the map $(w, i) \mapsto [w, i]$ is a bijection between $(T^{(1)})^* \times \{1, \ldots, r\}$ and $T^{(0)}$.

The following should be clear:

(1) If $u, u' \in T^{(1)}$, then $\mathbf{A} \models u \approx u'$ if and only if $u = u'$.
(2) If $[w, i], [w', j] \in T^{(0)}$, then $\mathbf{A} \models [w, i] \approx [w', j]$ if and only if $i = j$ and $G \models w \approx w'$.

Since each generator of $\mathbb{F}_n$ has finite order, the canonical map $(T^{(1)})^* \to \mathbb{F}_n$ is surjective. The claimed description of $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(r, n)$ follows from these remarks. □

Recall that G is fixed. Given $n > 0$ let P be the group $P_G(n, n{+}1)$ defined in Section 2, generated by $\{x_1, \ldots, x_{n+1}\}$ and with multiplication denoted by $*$. Define $\mathbf{P}(n) = \langle P, P, *\rangle$ and let $\mathbf{A}(n)$ be the subalgebra of $\mathbf{P}(n)$ having universes P and $\{x_1, \ldots, x_{n+1}\}$.

Lemma 4.2. *For all $n > 0$:*

(1) *$\mathbf{A}(n)$ is $(1, n{+}1)$-generated.*
(2) *$\mathbf{A}(n)$ is infinite if G is not nilpotent.*
(3) *Every (n, n)-generated subalgebra of $\mathbf{A}(n)$ can be embedded in $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n, n)$.*

Proof. Again let $P = P_G(n, n{+}1)$. Item 1 is obvious, while item 2 follows from Theorem 2.4. To prove item 3, suppose $\mathbf{C} = \langle C^{(0)}, C^{(1)}, *\rangle$ is an (n,n)-generated subalgebra of $\mathbf{A}(n)$ and let $s = |C^{(1)}|$. Then $s \le n$. Let H be the subsemigroup of the group P generated by $C^{(1)}$. Then H is a group, and in fact $H \cong \mathbb{F}_{\mathcal{V}(G)}(s)$ via an isomorphism sending $C^{(1)}$ to $\{x_1, \dots, x_s\}$. $C^{(0)} \subseteq P$ and H acts on $C^{(0)}$ by left multiplication. Thus each orbit of this action is a right coset of H in P, hence is isomorphic (as a left H-set) to $\mathbb{F}_{\mathcal{V}(G)}(s)$. Let r be the number of orbits in $C^{(0)}$ under this action; thus $r \le n$. It follows from Lemma 4.1 and the above remarks that $\mathbf{C} \cong \mathbb{F}_{\mathcal{V}(\mathbf{A})}(r, s)$. Hence $\mathbf{C}$ can be embedded in $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n, n)$. □

Theorem 4.3. *Suppose $b : G \times S \to S$ is a faithful left action of the finite group G on the finite set S, and $\mathbf{A}$ is the 2-sorted algebra of type τ constructed from b (as described in this section). Let $\mathbf{A}^*$ be the corresponding 1-sorted* L*-algebra defined in Section 3.*

(1) *If G is nilpotent, then $\mathcal{V}(\mathbf{A}^*)$ is finitely based.*
(2) *Suppose G is not nilpotent. If $\mathcal{K}$ is any class of* L*-algebras satisfying $\mathcal{V}(\mathbf{A}^*) = \mathcal{V}(\mathcal{K})$, then $\mathcal{K}$ is not contained in any finitely based, locally finite quasivariety.*

Proof. Item 1 follows from Lemma 4.1 and Corollary 2.2. To prove item 2, suppose $\mathcal{V}(\mathbf{A}^*) = \mathcal{V}(\mathcal{K})$ and $\mathcal{K} \subseteq \mathcal{Q}$, where $\mathcal{Q}$ is some quasivariety of L-algebras. Then $\mathbb{F}_{\mathcal{V}(\mathbf{A}^*)}(n) \in \mathcal{Q}$ for all $n > 0$. Now suppose that $\mathcal{Q}$ is finitely based. Then there exists an $n > 0$ such that, for any L-algebra $\mathbf{B}$, if every n-generated subalgebra of $\mathbf{B}$ is in $\mathcal{Q}$ then $\mathbf{B} \in \mathcal{Q}$. Consider the 2-sorted algebra $\mathbf{A}(n)$ constructed above. $\mathbf{A}(n)$ is infinite and finitely generated by Lemma 4.2, hence the corresponding L-algebra $\mathbf{A}(n)^*$ has these same properties. We shall show that $\mathbf{A}(n)^* \in \mathcal{Q}$, which will prove that $\mathcal{Q}$ is not locally finite.

Let $\mathbf{D}$ be an n-generated subalgebra of $\mathbf{A}(n)^*$. Then $\mathbf{D} = \mathbf{C}^*$ for some (n,n)-generated subalgebra $\mathbf{C}$ of $\mathbf{A}(n)$. By Lemma 4.2, $\mathbf{C}$ can be embedded in $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n, n)$. Hence $\mathbf{D}$ can be embedded in $\mathbb{F}_{\mathcal{V}(\mathbf{A})}(n, n)^* = \mathbb{F}_{\mathcal{V}(\mathbf{A}^*)}(n) \in \mathcal{Q}$, which proves $\mathbf{D} \in \mathcal{Q}$. By our choice of n, this shows $\mathbf{A}(n)^* \in \mathcal{Q}$. □

Note in particular that if $G = S_3$ and b is the faithful representation of S_3 as the set of permutations on $\{1, 2, 3\}$, then the algebra $\mathbf{A}^*$ described in Theorem 4.3 has 18 elements. Also note that $\mathbf{A}^*$ is 2-step strongly solvable (see a remark at the end of Section 3). Thus Theorem 4.3 strengthens one of the main results in [6].

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

E-mail address, Second author: `rdwillar@gillian.math.uwaterloo.ca`