A FINITE BASIS THEOREM FOR DIFFERENCE-TERM VARIETIES WITH A FINITE RESIDUAL BOUND

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Abstract. We prove that if $\mathcal{V}$ is a variety (i.e., an equationally axiomatizable class of algebraic structures) in a finite language, $\mathcal{V}$ has a difference term, and $\mathcal{V}$ has a finite residual bound, then $\mathcal{V}$ is finitely axiomatizable. This provides a common generalization of R. McKenzie's finite basis theorem for congruence modular varieties with a finite residual bound, and the R. Willard's finite basis theorem for congruence meet-semidistributive varieties with a finite residual bound.

1. Introduction

In [1], K. Baker proved the following theorem: if $\mathbf{A}$ is a finite algebra in a finite language, and the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ is congruence distributive, then the identities of $\mathbf{A}$ have a *finite basis* (i.e., the equational theory of $\mathbf{A}$ is finitely axiomatizable). Two important ingredients in the proof were provided by B. Jónsson [8]: (1) an explicit Maltsev characterization of the condition that a variety be congruence distributive, and (2) a proof that if $\mathbf{A}$ is finite and $\mathcal{V}(\mathbf{A})$ is congruence distributive, then every subdirectly irreducible member of $\mathcal{V}(\mathbf{A})$ has size at most $|A|$.

A variety is said to *have a finite residual bound* if there exists $r < \omega$ such that every subdirectly irreducible member of the variety has size at most r. In the mid-1970s Jónsson, Baker and possibly others wondered whether the existence of a finite residual bound, or some weaker hypothesis, might always imply that a variety is finitely based.[1] To the best of our knowledge, this speculation was first committed to print in the PhD thesis of Robert E. Park (1976), a student of Baker's. In his thesis,

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[1] In particular, see the report [23] from an Oberwolfach workshop in 1976. In the abstract of his talk (p. 1), Baker mentions "the conjecture of Jónsson that states that if a variety contains only finitely many subdirectly irreducible members, all finite, then it must be finitely definable," while in the Problems section (p. 28, Problem 39) Jónsson simply poses the following question: "Is it true for every variety $\mathcal{V}$ of algebras that if the class $\mathcal{V}_{\mathrm{FSI}}$ of all finitely subdirectly irreducible algebras of $\mathcal{V}$ is strictly elementary, then $\mathcal{V}$ is finitely based?" Finally, R. McKenzie writes ten years later [19, p. 226] that Jónsson "wondered, in the early 1970's, whether it is the case that every finite algebra $\mathbf{A}$ belonging to a residually small variety of finite type has a finite equational base."

Park examined five finite algebras which were known at that time to be not finitely based, proved that in each case the variety generated by the algebra does not have a finite residual bound, and then stated the conjecture which now bears his name:

Park's Conjecture [20, p. 89]: If a variety in a finite language has a finite residual bound, then the variety is finitely based.

Park's conjecture remains open to this day. Baker's theorem establishes the conjecture for congruence distributive varieties. R. McKenzie gave a significant generalization of Baker's theorem in 1987 when he verified Park's conjecture for congruence modular varieties [19]. R. Willard, in 2000, extended Baker's theorem in a different direction by confirming Park's conjecture for congruence meet-semidistributive varieties [22].

In this paper we verify Park's conjecture for a class of varieties which includes both congruence modular varieties and congruence meet-semidistributive varieties.

Definition 1.1. A term $p(x, y, z)$ is a *difference term* for a variety $\mathcal{V}$ if

(i) $\mathcal{V}$ satisfies the identity $p(x, x, y) \approx y$, and
(ii) $\mathcal{V}$ satisfies the property $p^{\mathbf{A}}(a, b, b) \equiv a \pmod{[\alpha, \alpha]}$ for all $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, where $\alpha = \mathrm{Cg}^{\mathbf{A}}(a, b)$ and $[\alpha, \alpha]$ denotes the usual ("term-condition") commutator of α with itself.

Here we note that [4] defines a "difference term" by *three* properties, namely (i) and (ii) above along with a third property we will not introduce. It can be shown that a variety has a term satisfying all three properties if and only if it is congruence modular, and for congruence modular varieties the third property is equivalent to the conjunction of properties (i) and (ii) of Definition 1.1. For congruence modular varieties the last term in H.-P. Gumm's Maltsev condition for congruence modularity is always a difference term [5, p. 53].

However, there are nonmodular varieties satisfying properties (i) and (ii) of Definition 1.1, and we follow [14] in defining a difference term with only these two properties. For example, the fact that $[\alpha, \alpha] = \alpha$ in congruence meet-semidistributive varieties, [13, Corollary 4.7], implies that the term $p(x, y, z) := z$ satisfies (i) and (ii) for such varieties; hence (for us) the third projection operation is a difference term for any congruence meet-semidistributive variety. The important point to remember is that the class of varieties with a difference term includes all congruence modular varieties and all congruence meet-semidistributive varieties.

Varieties with a difference term have been studied in [10, 13, 16, 17, 18], and have been revealed to be a reasonably natural class of varieties. In such varieties the commutator operation satisfies some (though not all) of the desirable features of the modular commutator. Additionally, K. Kearnes in [10] proved that a locally finite variety has a difference term if and only if it omits type **1** and is such that all type-**2** minimal sets have an empty tail. Kearnes and Á. Szendrei in [13] implicitly gave a

Maltsev condition characterizing the existence of a difference term, which formally resembles "congruence meet-semidistributive + permutable" in the same way that Gumm's Maltsev condition for congruence modularity resembles "congruence distributive + permutable." Combining the Maltsev condition of Kearnes and Szendrei with arguments from [22] in the obvious way yields:

Lemma 1.2. *Suppose $\mathcal{V}$ is a variety and $p(x, y, z)$ is a term. p is a difference term for $\mathcal{V}$ iff there exist 3-ary terms f_i, g_i ($i \in I$, I a finite set) satisfying the following conditions throughout $\mathcal{V}$:*

(1) $f_i(x, x, x) \approx x \approx g_i(x, x, x)$ *for all* $i \in I$.
(2) $f_i(x, y, x) \approx g_i(x, y, x)$ *for all* $i \in I$.
(3) $p(x, x, y) \approx y$.
(4) $\bigwedge_{i \in I}[f_i(a, a, b) = g_i(a, a, b) \leftrightarrow f_i(a, b, b) = g_i(a, b, b)] \ \rightarrow \ p(a, b, b) = a$.

The purpose of this paper is to prove the following theorem.

Theorem 1.3. *If $\mathcal{V}$ is a variety in a finite language, $\mathcal{V}$ has a difference term, and $\mathcal{V}$ has a finite residual bound, then $\mathcal{V}$ is finitely based.*

2. Commutator properties of varieties with a difference term

In this section we gather some known commutator properties of varieties with a difference term.

Suppose $\mathbf{A}$ is an algebra and $\alpha, \beta, \theta, \delta \in \operatorname{Con} \mathbf{A}$. $C(\alpha, \beta; \theta)$ is the usual ("term-condition") centralizer relation (see e.g. [7, Definition 3.3]), while $[\alpha, \beta]$ denotes the least $\gamma \in \operatorname{Con} \mathbf{A}$ satisfying $C(\alpha, \beta; \gamma)$. More generally, if $\delta \leq \alpha \wedge \beta$, then $[\alpha, \beta]_\delta$ denotes the least $\gamma \geq \delta$ such that $C(\alpha, \beta; \gamma)$; this equals the unique congruence $\gamma \geq \delta$ satisfying $\gamma/\delta = [\alpha/\delta, \beta/\delta]$, where the last commutator is calculated in $\mathbf{A}/\delta$. If $\gamma \leq \beta$, then $(\gamma : \beta)$ denotes the greatest $\alpha \geq \gamma$ satisfying $C(\alpha, \beta; \gamma)$; this equals the unique congruence $\alpha \geq \gamma$ satisfying $\alpha/\gamma = (0_{A/\gamma} : \beta/\gamma)$. If $a, b \in A$, we also write $\operatorname{ann}(a, b)$ for $(0_A : \operatorname{Cg}^{\mathbf{A}}(a, b))$.

Lemma 2.1. *Let $\mathcal{V}$ be a variety with a difference term p, and suppose $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.*

(1) $[\alpha, \beta] = [\beta, \alpha]$.
(2) *If α is abelian, then p is a Maltsev operation on each α-block.*

Proof. (1) is proved in [10, Lemma 2.2]. (2) follows directly from the definition of a difference term. □

Definition 2.2. Fix an algebra $\mathbf{A}$ and $a, b, c, d \in A$. Let $\alpha = \operatorname{Cg}^{\mathbf{A}}(a, b)$ and $\beta = \operatorname{Cg}^{\mathbf{A}}(c, d)$.

(1) $C(a, b, c, d)$ denotes the condition $C(\alpha, \beta; 0_A)$.

(2) $C_2(a,b,c,d)$ denotes the *two-term condition* for (α,β). That is, $C_2(a,b,c,d)$ iff for all $m,n\geq 1$, all $r_1,r_2\in \mathrm{Pol}_{m+n}(\mathbf{A})$, all $(a_1,b_1),\dots,(a_m,b_m)\in\alpha$, and all $(c_1,d_1),\dots,(c_n,d_n)\in\beta$, if three of the pairs

$$(r_1(\mathbf{a},\mathbf{c}),r_2(\mathbf{a},\mathbf{c})),\ (r_1(\mathbf{a},\mathbf{d}),r_2(\mathbf{a},\mathbf{d})),\ (r_1(\mathbf{b},\mathbf{c}),r_2(\mathbf{b},\mathbf{c})),\ (r_1(\mathbf{b},\mathbf{d}),r_2(\mathbf{b},\mathbf{d}))$$

are in 0_A, then so is the fourth.

Lemma 2.3. *If $\mathcal{V}$ has a difference term, then for all $\mathbf{A}\in\mathcal{V}$ and all $a,b,c,d\in A$, $C(a,b,c,d) \iff C_2(a,b,c,d)$.*

Proof. Follows from [13, Corollary 4.5] and Lemma 2.1(1). □

Lemma 2.4. *Suppose $\mathcal{V}$ is a variety with a difference term, $\mathbf{A}\in\mathcal{V}$, and $\alpha,\beta,\delta\in$* $\mathrm{Con}\,\mathbf{A}$.

(1) *$[\alpha,\beta]=0_A$ iff $[\mathrm{Cg}^{\mathbf{A}}(a,b),\mathrm{Cg}^{\mathbf{A}}(c,d)]=0_A$ for all $(a,b)\in\alpha$ and $(c,d)\in\beta$.*
(2) *If $\delta\leq\alpha\wedge\beta$, then $[\alpha,\beta]_\delta=[\alpha,\beta]\vee\delta$.*
(3) *If α is abelian, then $[\alpha,\beta]\leq\gamma$ iff $C(\beta,\alpha;\gamma)$.*

Proof. (1) follows in the usual way from left semi-distributivity and symmetry. (2) is [10, Lemma 2.4]. (3) follows from [10, Lemma 2.3] and symmetry. □

We also need the following fact from tame congruence theory.

Lemma 2.5. *Suppose $\mathbf{A}$ is a finite algebra and θ/δ and θ'/δ' are perspective prime quotients in* $\mathrm{Con}\,\mathbf{A}$ *with* $\mathrm{typ}(\delta,\theta)\neq\mathbf{1}$*. Then $(\delta:\theta)=(\delta':\theta')$.*

Proof. If θ/δ is non-abelian, then the claim follows from [7, Remark 5.13]. Assume $\mathrm{typ}(\delta,\theta)=\mathbf{2}$. Let U be a (δ,θ)-minimal set with $U=e(\mathbf{A})$ for some $e\in\mathrm{Pol}_1(\mathbf{A})$ satisfying $e^2=e$. As perspective prime quotients have the same minimal sets [7, Lemma 6.2], it suffices to show that $(\delta:\theta)$ has an intrinsic characterization referencing only $\mathbf{A}$ and U. Define

$$\begin{aligned}\gamma \;=\; \{(a,b)\in A^2 : \forall f\in\mathrm{Pol}_2(\mathbf{A}),\ & ef(a,x){\restriction}_U \text{ is a permutation}\\ \iff\ & ef(b,x){\restriction}_U \text{ is a permutation}\}.\end{aligned}$$

γ is clearly an equivalence relation and is invariant under unary polynomials of $\mathbf{A}$, so is a congruence. Let N be a (δ,θ)-trace in U. The proof of [12, Lemma 3.4 Case 2] shows $C(\gamma,N^2;\delta)$, which implies $C(\gamma,\theta;\delta)$ by [9, Lemma 4.2]; hence $\gamma\leq(\delta:\theta)$. Conversely, if $(\delta:\theta)\not\leq\gamma$, choose $(a,b)\in(\delta:\theta)\setminus\gamma$. By definition there is $f\in\mathrm{Pol}_2(\mathbf{A})$ such that $ef(a,x){\restriction}_U$ is a permutation of U and $ef(b,x){\restriction}_U$ is not (or the same with a and b interchanged). Let 0 and 1 be elements of N that are not δ related. Then (by properties of minimal sets) $(ef(a,0),ef(a,1))\in\delta$ while $(ef(b,0),ef(b,1))\notin\delta$, which proves $C((\delta:\theta),\theta;\delta)$ fails, which is impossible. □

Remark. Lemma 2.5 was proved in the case $U=A$ in [15, Theorem 3.4].

Finally, we need a fact about abelian principal congruences.

Definition 2.6. Given a variety $\mathcal{V}$ with difference term p, $\mathbf{A} \in \mathcal{V}$, $a, b \in A$, and $r > 0$, let $\Gamma_r(a,b) = \{(u, p(t(a,\mathbf{e}), t(b,\mathbf{e}), u)) : t \in \mathrm{Clo}_{r+1}(\mathbf{A}),\ \mathbf{e} \in A^r,\ u \in A\}$.

Lemma 2.7. *Suppose $\mathcal{V}$ is a variety with difference term p, $\mathbf{A} \in \mathcal{V}$, $a, b \in A$, and $r > 0$. If $\mathrm{Cg}^{\mathbf{A}}(a,b)$ is abelian and $\mathrm{ann}(a,b)$ has index at most r, then*
  (1) *$\mathrm{Cg}^{\mathbf{A}}(a,b) = \Gamma_r(a,b)$.*
  (2) *Each block of $\mathrm{Cg}^{\mathbf{A}}(a,b)$ has size at most $|\mathbb{F}_{\mathcal{V}}(r+1)|$.*

Proof. For (1), we mimic McKenzie's proof of the same claim in the congruence modular case [19, Lemma 2.16]. Start with the observation that the set θ of all pairs $(f(a), f(b))$, where f is a unary polynomial of $\mathbf{A}$, is a reflexive compatible relation on $\mathbf{A}$. The facts that $\mathrm{Cg}^{\mathbf{A}}(a,b)$ is abelian and that p is Maltsev on classes of abelian congruences implies that θ is a congruence. Hence if $(u,v) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$, then there exists a polynomial $f(x) = s(x,\mathbf{e})$, $s \in \mathrm{Clo}(\mathbf{A})$, such that

$$(v,u) = (s(a,\mathbf{e}), s(b,\mathbf{e})).$$

Then

$$p(s(a,\underline{\mathbf{e}}), s(b,\underline{\mathbf{e}}), u) = p(v,u,u) = v = p(v,v,v) = p(s(a,\underline{\mathbf{e}}), s(a,\underline{\mathbf{e}}), v).$$

Since $(a,b),(u,v) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$, we retain the equality of left and right hand sides if we simultaneously change all underlined occurrences of $\mathbf{e}$ to any tuple $\mathbf{e}'$ that is congruent to $\mathbf{e}$ modulo $\mathrm{ann}(a,b)$ coordinatewise. Thus

$$p(s(a,\underline{\mathbf{e}}'), s(b,\underline{\mathbf{e}}'), u) = p(s(a,\underline{\mathbf{e}}'), s(a,\underline{\mathbf{e}}'), v) = v,$$

where the last equality follows from the identity $p(x,x,y) \approx y$. We may choose $\mathbf{e}'$ so that it has at most $|1/\mathrm{ann}(a,b)| = r$ distinct entries, and write $\mathbf{e}''$ for a sequence of length r containing the distinct entries of $\mathbf{e}'$. There is a $t \in \mathrm{Clo}_{r+1}(\mathbf{A})$ such that $s(x,\mathbf{e}') = t(x,\mathbf{e}'')$ holds for all $x \in A$. This shows that

$$(u,v) = (u, p(t(a,\underline{\mathbf{e}}''), t(b,\underline{\mathbf{e}}''), u))$$

for some $t \in \mathrm{Clo}_{r+1}(\mathbf{A})$, $\mathbf{e}'' \in A^r$, $u \in A$. Hence $(u,v) \in \Gamma_r(a,b)$. Since $(u,v) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$ was arbitrary, we get that $\mathrm{Cg}^{\mathbf{A}}(a,b) \subseteq \Gamma_r(a,b)$.

The reverse inclusion, $\Gamma_r(a,b) \subseteq \mathrm{Cg}^{\mathbf{A}}(a,b)$, is an immediate consequence of the identity $p(x,x,y) \approx y$. Specifically, if $(u,v) = (u, p(t(a,\underline{\mathbf{e}}), t(b,\underline{\mathbf{e}}), u)) \in \Gamma_r(a,b)$, then

$$v = p(t(a,\mathbf{e}), t(b,\mathbf{e}), u) \overset{\mathrm{Cg}(a,b)}{\equiv} p(t(a,\mathbf{e}), t(a,\mathbf{e}), u) = u.$$

(2) follows from (1) and the fact that we can choose one fixed $\mathbf{e}$ in the definition of $\Gamma_r(a,b)$ (namely, a transversal for $\mathrm{ann}(a,b)$). Then for any $u \in A$ the function

$$t \mapsto p(t(a,\mathbf{e}), t(b,\mathbf{e}), u)$$

maps $\mathrm{Clo}_{r+1}(\mathbf{A})$ surjectively onto the $\mathrm{Cg}^{\mathbf{A}}(a,b)$-block of u. $\square$

3. The commutator identity C1

C1 is the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$, or equivalently, the implication $\alpha \leq [\beta, \beta] \implies [\alpha, \beta] = \alpha$. C1 was identified in [3] and named in [4]. In this section we collect the facts about C1 that we will need.

Proposition 3.1. *Suppose $\mathcal{V}$ is a locally finite variety with a difference term.*

(1) *$\mathcal{V}$ satisfies* C1 *if and only if $(0_A : \mu)$ is abelian for every finite subdirectly irreducible algebra $\mathbf{A} \in \mathcal{V}$ with abelian monolith μ.*
(2) *If $\mathcal{V}$ is residually small, then $\mathcal{V}$ satisfies* C1.

Proof sketch. (1) The ($\Rightarrow$) implication is proved by applying C1 to the situation $\alpha = \mu$ and $\beta = (0_A : \mu)$.

For ($\Leftarrow$), assume that C1 fails. Then it fails in a finite algebra $\mathbf{A} \in \mathcal{V}$; say $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ with $\alpha \leq \beta$ and $[\alpha, \beta] < \alpha \wedge [\beta, \beta]$. Because $[\alpha, \beta]$ lies below all the relevant congruences and commutators in this witnessing failure, we can factor by $[\alpha, \beta]$ and obtain a parallel failure of C1 in $\mathbf{A}/[\alpha, \beta]$ (using Lemma 2.4(2)). Thus we may assume that $[\alpha, \beta] = 0_A$ and $\alpha \wedge [\beta, \beta] > 0_A$. Choose an atom γ below $\alpha \wedge [\beta, \beta]$. Let δ be a completely meet irreducible that is disjoint from γ, but whose upper cover θ contains γ. Then $(0_A : \gamma) = (\delta : \theta) =: \psi$ by Lemma 2.5, and $\beta \leq \psi$ because $[\beta, \alpha] = [\alpha, \beta] = 0_A$ and $\gamma \leq \alpha$. Then in $\mathbf{A}/\delta$, $(0_{A/\delta} : \theta/\delta) = (\delta : \theta)/\delta = \psi/\delta$, while Lemma 2.4(2) yields $[\psi/\delta, \psi/\delta] = ([\psi, \psi] \vee \delta)/\delta \neq 0_{A/\delta}$ as $[\beta, \beta] \not\leq \delta$ and $\beta \leq \psi$. Hence $\mathbf{A}/\delta$ violates the conclusion of (1).

(2) follows from (1) and [11, Corollary 4.3]. □

Proposition 3.2. *Suppose $\mathcal{V}$ is a locally finite variety, $\mathcal{V}$ omits type $\mathbf{1}$, $\mathcal{V}$ satisfies* C1*, and there exists a positive integer r such that for every finite subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ with monolith μ, $(0_A : \mu)$ has index at most r. Then $\mathcal{V}$ has a finite residual bound.*

Proof. Let $m = |\mathbb{F}_{\mathcal{V}}(r+1)|$. It will suffice by Quackenbush's Theorem [21] to prove that every *finite* subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ has size at most $r \cdot m^m$. Let μ be the monolith of $\mathbf{A}$ and let $\alpha = (0_A : \mu)$. The claim follows immediately if μ is nonabelian, so assume μ is abelian. Then $\operatorname{typ}(0_A, \mu) = \mathbf{2}$, and α is abelian by C1 and Proposition 3.1(1). The proof of [11, Theorem 5.1] shows that each class of α has size at most m^m, which proves the claim. (Alternatively, one can mimic the proof of [3, Theorem 8] to show that each class of α has size at most $(m+1)!$.) □

The next result is inspired by McKenzie's proof of his finite basis theorem [19].

Proposition 3.3. *Let $\mathcal{V}$ be a locally finite variety with a difference term. $\mathcal{V}$ satisfies* C1 *if and only if for all (or all finite) $\mathbf{A} \in \mathcal{V}$, all of the following conditions hold:*

(1) *If $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ are abelian, then $\alpha \vee \beta$ is abelian.*
(2) *If β is a principal congruence of $\mathbf{A}$ and $[\beta, [\beta, \beta]] = 0_A$, then β is abelian.*

(3) *If* $\alpha_0, \alpha_1, \beta_1, \beta_2 \in \operatorname{Con}\mathbf{A}$ *with* β_1, β_2 *principal,* $0_A \prec \alpha_0 \prec \alpha_1$*,* α_1 *abelian, and* $[\alpha_0, \beta_1] = [\alpha_0, \beta_2] = 0_A$*, then there exists an abelian atom* $\gamma \in \operatorname{Con}\mathbf{A}$ *such that* $[\alpha_1, \beta_1], [\alpha_1, \beta_2] \leq \gamma$*.*

(4) *If* $0_A \prec \alpha \prec \beta$ *in* $\operatorname{Con}\mathbf{A}$ *with* α *abelian and* $[\beta, \beta] = \beta$*, then* $[\alpha, \beta] = \alpha$*.*

Proof. ($\Rightarrow$) Assume $\mathcal{V}$ satisfies C1 and $\mathbf{A} \in \mathcal{V}$. To prove (1), it suffices by semi-distributivity and symmetry to prove $[\alpha, \beta] = 0_A$. Let $\delta = [\alpha, \beta]$. Then $[\alpha, \delta] \leq [\alpha, \alpha] = 0_A$ and similarly $[\beta, \delta] = 0_A$. Thus if $\gamma = \alpha \vee \beta$ then $[\gamma, \delta] = [\delta, \gamma] = 0_A$ by semi-distributivity. But $\delta \leq [\gamma, \gamma]$, so with C1 this implies $\delta = 0_A$ as required. To prove (2), let $\alpha = [\beta, \beta]$ and apply the implicational version of C1. To prove (3), note first that the hypotheses give $[\alpha_0, \beta_1 \vee \beta_2] = 0_A$. Assume next that $[\alpha_1, \beta_1 \vee \beta_2]$ is not at height 0 or 1 in $\operatorname{Con}\mathbf{A}$. Note that as α_1 is abelian, the interval from 0_A to α_1 is solvable, hence is a modular sublattice of $\operatorname{Con}\mathbf{A}$ by [7, Lemma 6.5] and so has height 2. These facts imply $[\alpha_1, \beta_1 \vee \beta_2] = \alpha_1$. But then $\alpha_1 \leq [\beta_1 \vee \beta_2, \beta_1 \vee \beta_2]$, so C1 implies $\alpha_0 = [\alpha_0, \beta_1 \vee \beta_2]$, contrary to a previous calculation. Thus $\delta := [\alpha_1, \beta_1 \vee \beta_2]$ has height 0 or 1. If $\delta = 0_A$ then we can take $\gamma = \alpha_0$, while if $\delta \neq 0_A$ then we can take $\gamma = \delta$, which proves (3). (4) is an immediate consequence of C1.

($\Leftarrow$) Assume that every finite $\mathbf{A} \in \mathcal{V}$ satisfies (1)–(4) but $\mathcal{V}$ fails to satisfy C1. Then C1 fails in some finite member of $\mathcal{V}$. Let $\mathbf{A}$ be a finite member of $\mathcal{V}$ of minimum cardinality in which C1 fails, and pick $\alpha, \beta \in \operatorname{Con}\mathbf{A}$ with $\alpha \leq [\beta, \beta]$ but $[\alpha, \beta] < \alpha$. Note that, by (1), $\mathbf{A}$ has a largest abelian congruence which we will denote by ν.

We first prove ($*$) $[\theta, [\theta, \theta]] = [\theta, \theta]$ for all $\theta \in \operatorname{Con}\mathbf{A}$. Indeed, suppose $\delta := [\theta, [\theta, \theta]] < [\theta, \theta]$. Then in $\mathbf{A}/\delta$, $[\theta/\delta, \theta/\delta] = [\theta, \theta]/\delta \neq 0_{A/\delta}$ but $[\theta/\delta, [\theta/\delta, \theta/\delta]] = 0_{A/\delta}$, both by Lemma 2.4(2). Hence $\mathbf{A}/\delta$ fails to satisfy C1, so by minimality we have $\delta = 0_A$. Next observe that, since $[\theta, \theta] \neq 0_A$ we have $\theta \not\leq \nu$; pick $(a, b) \in \theta \setminus \nu$ and put $\beta' = \operatorname{Cg}^{\mathbf{A}}(a, b)$. Then β' violates condition (2).

In particular, $[\beta, \beta] = [[\beta, \beta], \beta]$. As $\alpha \leq [\beta, \beta]$ but $[\alpha, \beta] \neq \alpha$, there exist $\alpha \leq \alpha_0 \prec \alpha_1 \leq [\beta, \beta]$ such that $[\alpha_1, \beta] = \alpha_1$ but $[\alpha_0, \beta] < \alpha_0$. Because $[\alpha_0, \beta]$ is below all of the relevant congruences and commutators, we can factor by it and still preserve the above facts (by Lemma 2.4(2)); thus by minimality we have $[\alpha_0, \beta] = 0_A$. As $\alpha_0 \leq \alpha_1 = [\alpha_1, \beta] \leq \beta$, it follows that α_0 is abelian. Choose $0_A \leq \psi \prec \alpha_0$. Lemma 2.4(3) then gives $C(\beta, \alpha_0; \psi)$. Hence we can factor by ψ and preserve the relevant facts, so by minimality, $0_A \prec \alpha_0$.

In summary, we have $0_A \prec \alpha_0 \prec \alpha_1$, $[\alpha_1, \beta] = \alpha_1$, and $[\alpha_0, \beta] = 0_A$. This implies $\alpha_1 \leq \beta$, so $\alpha_0 \leq [\beta, \beta]$. The proof of Proposition 3.1(1) then shows $\mathbf{A}$ is subdirectly irreducible, by minimality, so α_0 is its monolith. Consider $\mu := [\alpha_1, \alpha_1]$. By the fact ($*$) established two paragraphs back (with $\theta = \alpha_1$), $[\alpha_1, \mu] = \mu$, which with $[\alpha_0, \alpha_1] = 0_A$ implies $\mu \neq \alpha_0$. Hence $\mu \in \{0_A, \alpha_1\}$.

Case 1: $\mu = 0_A$, i.e., α_1 is abelian.

Assume that β is minimal among all congruences β' satisfying $[\alpha_0, \beta'] = 0_A$ and $[\alpha_1, \beta'] = \alpha_1$. As $[\beta, \alpha_1] = \alpha_1$ we have $\neg C(\beta, \alpha_1; 0_A)$ and $\neg C(\beta, \alpha_1; \alpha_0)$. We can pick principal congruences $\beta_1, \beta_2 \le \beta$ witnessing $\neg C(\beta_1, \alpha_1; 0_A)$ and $\neg C(\beta_2, \alpha_1; \alpha_0)$. Then $\beta' := \beta_1 \vee \beta_2$ satisfies $[\alpha_0, \beta'] = 0_A$ and $[\alpha_1, \beta'] = \alpha_1$, so $\beta = \beta'$.

By condition (3) and subdirectly irreducible, we have $[\alpha_1, \beta_i] \le \alpha_0$ for $i = 1, 2$. If $[\alpha_1, \beta_1] = [\alpha_1, \beta_2]$ then $[\alpha_1, \beta] \le \alpha_0$ by semi-distributivity, contradicting $[\alpha_1, \beta] = \alpha_1$. Hence $[\alpha_1, \beta_1] = \alpha_0$ and $[\alpha_1, \beta_2] = 0_A$. Hence $C(\beta_1, \alpha_1; \alpha_0)$, and as α_1 is abelian, we get $C(\beta_2, \alpha_1; \alpha_0)$ by Lemma 2.4(3). Thus $C(\beta, \alpha_1; \alpha_0)$, which contradicts $[\alpha_1, \beta] = \alpha_1$. This case is impossible.

Case 2: $\mu = \alpha_1$.

Then we have a violation of condition (4) (with α, β replaced by α_0, α_1). This case is also impossible. □

4. Characterizing the principal centralizer relation

Our goal in this section is to provide characterizations of $C(a, b, c, d)$ in varieties with a difference term, similar to the characterizations of $C(a, b, c, d)$ in congruence modular varieties provided in [19, Theorem 2.7] and [4, Chapter 6, Exercise 6].

Definition 4.1. Fix an algebra $\mathbf{A}$ and $a, b, c, d \in A$.

(1) If $r \in \mathrm{Pol}_2(\mathbf{A})$, $\vec{H}^r(a, b, c, d)$ is the implication $r(a, c) = r(a, d) \implies r(b, c) = r(b, d)$.

(2) $\vec{H}(a, b, c, d)$ iff $\vec{H}^r(a, b, c, d)$ for all $r \in \mathrm{Pol}_2(\mathbf{A})$.

(3) $H(a, b, c, d)$ iff $\vec{H}(a, b, c, d)$ & $\vec{H}(b, a, c, d)$.

Definition 4.2. Fix an algebra $\mathbf{A}$ and $a, b, c, d \in A$.

(1) If $r_1, r_2 \in \mathrm{Pol}_2(\mathbf{A})$, $\vec{H}_2^{r_1, r_2}(a, b, c, d)$ is the implication

$$[r_1(a, c) = r_2(a, c) \ \& \ r_1(a, d) = r_2(a, d) \ \& \ r_1(b, c) = r_2(b, c)] \implies r_1(b, d) = r_2(b, d).$$

The pair $(r_1(b, d), r_2(b, d))$ is called the *critical pair* of $\vec{H}_2^{r_1, r_2}(a, b, c, d)$ (whether $\vec{H}_2^{r_1, r_2}(a, b, c, d)$ holds or not).

(2) $\vec{H}_2(a, b, c, d)$ iff $\vec{H}_2^{r_1, r_2}(a, b, c, d)$ for all $r_1, r_2 \in \mathrm{Pol}_2(\mathbf{A})$.

(3) $H_2(a, b, c, d)$ iff $\vec{H}_2(a, b, c, d)$ & $\vec{H}_2(b, a, c, d)$ & $\vec{H}_2(a, b, d, c)$ & $\vec{H}_2(b, a, d, c)$.

Lemma 4.3. *For any algebra $\mathbf{A}$ and $a, b, c, d \in A$:*

(1) *$C_2(a, b, c, d)$ implies $H_2(a, b, c, d)$.*

(2) *$H_2(a, b, c, d)$ implies $H(a, b, c, d)$.*

The next condition is borrowed from McKenzie's relation K [19].

Definition 4.4. Let $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$.

(1) If $r \in \mathrm{Pol}_2(\mathbf{A})$, $\vec{K}_p^r(a, b, c, d)$ is the following equation:

$$p(r(a, c), r(b, c), r(b, d)) = p(r(a, d), r(b, d), r(b, d)).$$

(2) $\vec{K}_p(a, b, c, d)$ iff $\vec{K}_p^r(a, b, c, d)$ for all $r \in \mathrm{Pol}_2(\mathbf{A})$.

(3) $K_p(a,b,c,d)$ iff $\vec{K}_p(a,b,c,d)$ & $\vec{K}_p(a,b,d,c)$ & $\vec{K}_p(b,a,c,d)$ & $\vec{K}_p(b,a,d,c)$.

Lemma 4.5. *Suppose $\mathcal{V}$ is a variety with a difference term, $\mathbf{A} \in \mathcal{V}$, and $a,b,c,d \in A$. Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$ and $\beta = \mathrm{Cg}^{\mathbf{A}}(c,d)$.*

(1) *$\vec{H}(a,b,c,d)$ implies $\vec{K}_p(a,b,c,d)$.*

(2) *Suppose $r_1, r_2 \in \mathrm{Pol}_2(\mathbf{A})$ and define*

$$\widehat{r}_2(x,y) = r_2(y,x) \quad \text{and} \quad s(x,y) = p(r_1(x,y), r_2(x,d), r_2(b,d)).$$

Suppose $\vec{K}_p^{\widehat{r}_2}(c,d,a,b)$ and $\vec{K}_p^{s}(b,a,c,d)$ hold but $\vec{H}_2^{r_1,r_2}(a,b,c,d)$ fails with critical pair (u,v). Then $p(p(u,v,v),v,v) = v$.

(3) *If $K_p(a,b,c,d)$ and $K_p(c,d,a,b)$ but $\neg \vec{H}_2(a,b,c,d)$, then $\alpha \cap \beta$ is not abelian.*

Proof. (1) Given $r \in \mathrm{Pol}_2(\mathbf{A})$, let $r'(x,y) = p(r(a,y), r(x,y), r(b,d))$. Then

$$\begin{aligned}
r'(a,c) &= p(r(a,c), r(a,c), r(b,d)) = r(b,d)\\
r'(a,d) &= p(r(a,d), r(a,d), r(b,d)) = r(b,d)\\
r'(b,c) &= p(r(a,c), r(b,c), r(b,d))\\
r'(b,d) &= p(r(a,d), r(b,d), r(b,d)).
\end{aligned}$$

As $r'(a,c) = r'(a,d)$, $\vec{H}^{r'}(a,b,c,d)$ implies $r'(b,c) = r'(b,d)$, which is $\vec{K}_p^{r}(a,b,c,d)$.

(2) As $\neg \vec{H}_2^{r_1,r_2}(a,b,c,d)$, we have

$$\begin{aligned}
r_1(a,c) &= r_2(a,c) =: C\\
r_1(b,c) &= r_2(b,c) =: D\\
r_1(a,d) &= r_2(a,d) := E\\
u = r_1(b,d) &\neq r_2(b,d) = v.
\end{aligned}$$

$\vec{K}_p^{\widehat{r}_2}(c,d,a,b)$ gives

$$p(r_2(a,c), r_2(a,d), r_2(b,d)) = p(r_2(b,c), r_2(b,d), r_2(b,d)),$$

i.e., $p(C,E,v) = p(D,v,v)$. This can be rewritten as

$$p(r_1(a,c), r_2(a,d), r_2(b,d)) = p(r_1(b,c), r_2(b,d), r_2(b,d)).$$

Observe that the last displayed equation can be written as $s(a,c) = s(b,c)$. $\vec{K}_p^{s}(b,a,c,d)$ gives

$$p(s(b,c), s(a,c), s(a,d)) = p(s(b,d), s(a,d), s(a,d)),$$

which by the last observation is equivalent to

$$s(a,d) = p(s(b,d), s(a,d), s(a,d)). \tag{4.1}$$

Calculating, we find

$$\begin{aligned}
s(a,d) &= p(r_1(a,d), r_2(a,d), r_2(b,d)) = p(E,E,v) = v\\
s(b,d) &= p(r_1(b,d), r_2(b,d), r_2(b,d)) = p(u,v,v).
\end{aligned}$$

Thus equation (4.1) gives $p(p(u,v,v),v,v) = v$.

(3) As the hypotheses are symmetric, we may assume that $\neg\vec{H}_2^{r_1,r_2}(a,b,c,d)$ fails for some $r_1, r_2 \in \mathrm{Pol}_2(\mathbf{A})$. Define $u = r_1(b,d)$ and $v = r_2(b,d)$, so $u \neq v$. By item (2), we have $p(p(u,v,v),v,v) = v$, which with $u \neq v$ implies $p(u,v,v) \neq u$. But $(u,v) \in \alpha \cap \beta$, so p is not a Maltsev operation on $(\alpha \cap \beta)$-blocks, so $\alpha \cap \beta$ is not abelian by Lemma 2.1(2). □

The next definition is the first of two which addresses the operations in Lemma 1.2. For the remainder of this section, if $\mathcal{V}$ is a variety with a difference term p, then we assume that f_i, g_i $(i \in I)$ is a finite family of ternary terms witnessing Lemma 1.2 for $\mathcal{V}, p$.

Definition 4.6. Let $\mathcal{V}$ be a variety with a difference term, $\mathbf{A} \in \mathcal{V}$ and $a,b,c,d \in A$.

(1) If $s, t \in \mathrm{Pol}_1(\mathbf{A})$ and $i \in I$, then $\vec{K}_{fg}^{s,t,i}(a,b,c,d)$ is the implication
$$\begin{aligned} &f_i(s(a),t(c),s(b)) = g_i(s(a),t(c),s(b)) \\ &\quad\Longrightarrow\quad f_i(s(a),t(d),s(b)) = g_i(s(a),t(d),s(b)). \end{aligned}$$
(2) $\vec{K}_{fg}(a,b,c,d)$ iff $\vec{K}_{fg}^{s,t,i}(a,b,c,d)$ for all $s,t \in \mathrm{Pol}_1(\mathbf{A})$ and all $i \in I$.

(3) $K_{fg}(a,b,c,d)$ iff $\vec{K}_{fg}(a,b,c,d)$ & $\vec{K}_{fg}(b,a,c,d)$ & $\vec{K}_{fg}(a,b,d,c)$ & $\vec{K}_{fg}(b,a,d,c)$.

The final relation to be defined generalizes K_{fg}, but is less well-behaved.

Definition 4.7. Let $\mathcal{V}$ be a variety with a difference term, $\mathbf{A} \in \mathcal{V}$ and $a,b,c,d \in A$.

(1) If $s_1, s_2, t \in \mathrm{Pol}_1(\mathbf{A})$ and $i \in I$, then $\vec{L}_{fg}^{s_1,s_2,t,i}(a,b,c,d)$ is the implication
$$\begin{aligned} &[s_1(a) = s_2(a) \quad \& \quad f_i(s_1(b),t(c),s_2(b)) = g_i(s_1(b),t(c),s_2(b)] \\ &\quad\Longrightarrow\quad f_i(s_1(b),t(d),s_2(b)) = g_i(s_1(b),t(d),s_2(b). \end{aligned}$$
(2) $\vec{L}_{fg}(a,b,c,d)$ iff $\vec{L}_{fg}^{s_1,s_2,t,i}(a,b,c,d)$ for all $s_1,s_2,t \in \mathrm{Pol}_1(\mathbf{A})$ and all $i \in I$.

(3) $L_{fg}(a,b,c,d)$ iff $\vec{L}_{fg}(a,b,c,d)$ & $\vec{L}_{fg}(a,b,d,c)$ & $\vec{L}_{fg}(b,a,c,d)$ & $\vec{L}_{fg}(b,a,d,c)$.

Lemma 4.8. *Let $\mathcal{V}$ be a variety with a difference term, $\mathbf{A} \in \mathcal{V}$ and $a,b,c,d \in A$.*

(1) *$\vec{H}_2(a,b,c,d)$ implies $\vec{L}_{fg}(a,b,c,d)$.*

(2) *$\vec{L}_{fg}(a,b,c,d)$ implies $\vec{K}_{fg}(a,b,c,d)$.*

Proof. (1) Given $s_1,s_2,t \in \mathrm{Pol}_1(\mathbf{A})$ and $i \in I$, define $r_1(x,y) = f_i(s_1(x),t(y),s_2(x))$ and $r_2(x,y) = g_i(s_1(x),t(y),s_2(x))$. Then $\vec{H}_2^{r_1,r_2}(a,b,c,d)$ implies $\vec{L}_{fg}^{s_1,s_2,t,i}(a,b,c,d)$.

(2) Given $s,t \in \mathrm{Pol}_1(\mathbf{A})$, define $s_2 = s$ and $s_1(x) = s(a)$. Then $\vec{L}_{fg}^{s_1,s_2,t,i}(a,b,c,d) \equiv \vec{K}_{fg}^{s,t,i}(a,b,c,d)$. □

Lemma 4.9. *Let $\mathcal{V}$ be a variety with a difference term, $\mathbf{A} \in \mathcal{V}$ and $a,b,c,d \in A$.*

(1) *$H(c,d,a,b)$ and $L_{fg}(a,b,c,d)$ imply $C(a,b,c,d)$.*

(2) *$K_p(a,b,c,d)$ and $K_{fg}(a,b,c,d)$ imply $H(c,d,a,b)$.*

Proof. (1) Assume $H(c,d,a,b)$ and $L_{fg}(a,b,c,d)$ hold but $C(a,b,c,d)$ fails. Thus there exist $(c_1,d_1),\dots,(c_n,d_n) \in \mathrm{Cg}^{\mathbf{A}}(c,d)$ and $r \in \mathrm{Pol}_{1+n}(\mathbf{A})$ such that, without loss of generality,

$$\begin{array}{rcrcl} & & r(a,\mathbf{c}) & = & r(a,\mathbf{d}) \\ c' & := & r(b,\mathbf{c}) & \neq & r(b,\mathbf{d}) \ =: \ d'. \end{array}$$

Define

$$r'(\mathbf{x},y) = p(r(y,\mathbf{x}), r(y,\mathbf{c}), r(b,\mathbf{c})).$$

Starting from $r'(\mathbf{c},a) = r'(\mathbf{c},b)$ and using $H(c,d,a,b)$ and Maltsev chains of polynomial images of $\{c,d\}$ connecting each c_i to d_i, we can deduce a succession of equations, the last of which is $r'(\mathbf{d},a) = r'(\mathbf{d},b)$, i.e.,

$$c' = p(d',c',c').$$

Since $c' \neq d'$, the difference term axioms give $i \in I$ such that

$$f_i(d',c',c') = g_i(d',c',c') \iff f_i(d',d',c') \neq g_i(d',d',c').$$

Assume with no loss of generality that $f_i(d',c',c') \neq g_i(d',c',c')$ while $f_i(d',d',c') = g_i(d',d',c')$. Define

$$\begin{array}{rcl} e_0 & = & r(b,c_1,c_2,\dots,c_n) \ = \ c' \\ e_1 & = & r(b,d_1,c_2,\dots,c_n) \\ & \vdots & \\ e_j & = & r(b,d_1,\dots,,d_j,c_{j+1},\dots,c_n) \\ & \vdots & \\ e_n & = & r(b,d_1,d_2,\dots,d_n) \ = \ d'. \end{array}$$

As $f_i(d',e_0,c') \neq g_i(d',e_0,c')$ but $f_i(d',e_n,c') = g_i(d',e_n,c')$, there exists $1 \leq j \leq n$ such that $f_i(d',e_{j-1},c') \neq g_i(d',e_{j-1},c')$ while $f_i(d',e_j,c') = g_i(d',e_j,c')$. Define

$$\begin{array}{rcl} \sigma_1(x) & = & r(x,\mathbf{d}) \\ \sigma_2(x) & = & r(x,\mathbf{c}) \\ t(x) & = & r(b,d_1,\dots,d_{j-1},x,c_{j+1},\dots,c_n). \end{array}$$

Observe that

$$\begin{array}{rcl} \sigma_1(a) & = & \sigma_2(a) \\ f_i(\sigma_1(b),t(c_j),\sigma_2(b)) & \neq & g_i(\sigma_1(b),t(c_j),\sigma_2(b)) \\ f_i(\sigma_1(b),t(d_j),\sigma_2(b)) & = & g_i(\sigma_1(b),t(d_j),\sigma_2(b)). \end{array}$$

As $(c_j,d_j) \in \mathrm{Cg}^{\mathbf{A}}(c,d)$, there exists a Maltsev chain $c_j = u_0,u_1,\dots,u_m = d_j$ and unary polynomials $\lambda_1,\dots,\lambda_m$ such that $\{\lambda_k(c),\lambda_k(d)\} = \{u_{k-1},u_k\}$ for each

k. Choose k such that

$$\begin{array}{rcl} f_i(\sigma_1(b), t(u_{k-1}), \sigma_2(b)) & \neq & g_i(\sigma_1(b), t(u_{k-1}), \sigma_2(b)) \\ f_i(\sigma_1(b), t(u_k), \sigma_2(b)) & = & g_i(\sigma_1(b), t(u_k), \sigma_2(b)). \end{array}$$

Let $\lambda(x) = t(\lambda_k(x))$. Then

$$\begin{array}{rl} & f_i(\sigma_1(b), \lambda(c), \sigma_2(b)) = g_i(\sigma_1(b), \lambda(c), \sigma_2(b)) \\ \iff & f_i(\sigma_1(b), \lambda(d), \sigma_2(b)) \neq g_i(\sigma_1(b), \lambda(d), \sigma_2(b)). \end{array}$$

This is a violation of $L_{fg}(a, b, c, d)$.

(2) Assume that $K_{fg}(a, b, c, d)$ and $K_p(a, b, c, d)$ hold but $\vec{H}(c, d, a, b)$ fails at $r_1 \in \mathrm{Pol}_2(\mathbf{A})$. Define $r(x, y) = r_1(y, x)$. Thus $r(a, c) = r(b, c)$ but $r(a, d) \neq r(b, d)$. Define

$$s(x) = r(x, d) \quad \text{and} \quad t(y) = r(a, y) \quad \text{and} \quad t'(y) = r(b, y).$$

Define

$$\begin{array}{rcccccl} B & = & r(a, c) & = & t(c) & & \\ B' & = & r(b, c) & = & t'(c) & & \\ A & = & r(a, d) & = & t(d) & = & s(a) \\ C & = & r(b, d) & = & t'(d) & = & s(b). \end{array}$$

Applying $K_{fg}(a, b, c, d)$ at s, t yields

$$f_i(A, B, C) = g_i(A, B, C) \leftrightarrow f_i(A, A, C) = g_i(A, A, C),$$

while applying $K_{fg}(a, b, c, d)$ at s, t' yields

$$f_i(A, B', C) = g_i(A, B', C) \leftrightarrow f_i(A, C, C) = g_i(A, C, C).$$

As $B = B'$, we get

$$f_i(A, A, C) = g_i(A, A, C) \leftrightarrow f_i(A, C, C) = g_i(A, C, C) \quad \text{for all } i \in I.$$

Hence $p(A, C, C) = A$ by the difference-term axioms. Now apply $K_p(a, b, c, d)$ at r to get

$$p(B, B', C) = p(A, C, C).$$

As $B = B'$, one of the difference term identities gives $p(A, C, C) = C$. This proves $A = C$. But that contradicts our assumptions. □

Corollary 4.10. *Let $\mathcal{V}$ be a variety with a difference term, $\mathbf{A} \in \mathcal{V}$, and $a, b, c, d \in A$. The following are equivalent:*

(1) *$C(a, b, c, d)$.*
(2) *$H_2(a, b, c, d)$.*
(3) *$K_p(a, b, c, d)$ and $L_{fg}(a, b, c, d)$.*

Proof. (1) $\Rightarrow$ (2). By Lemma 2.3 and Lemma 4.3(1).

(2) $\Rightarrow$ (3). $H_2(a,b,c,d) \Rightarrow H(a,b,c,d) \Rightarrow K_p(a,b,c,d)$ by Lemma 4.3(2) and Lemma 4.5(1). $H_2(a,b,c,d) \Rightarrow L_{fg}(a,b,c,d)$ by Lemma 4.8(1).

(3) $\Rightarrow$ (1). $K_p(a,b,c,d)$ and $L_{fg}(a,b,c,d)$ imply $H(c,d,a,b)$ by Lemma 4.9(2). This with $L_{fg}(a,b,c,d)$ implies $C(a,b,c,d)$ by Lemma 4.9(1). □

5. Definability in varieties with a finite residual bound

In this section we study the relation $C(a,b,c,d)$ in varieties with a difference term and having a finite residual bound. For this purpose, we introduce more notation.

Definition 5.1. If $k \geq 1$ and $m \geq 0$, then $\mathrm{Pol}_k^{(m)}(\mathbf{A})$ denotes the set of $r \in \mathrm{Pol}_k(\mathbf{A})$ which can realized by a term operation of $\mathbf{A}$ using at most m parameters from A; that is, $r(\mathbf{x}) = t(\mathbf{x}, \mathbf{e})$ for some $t \in \mathrm{Clo}_{k+m}(\mathbf{A})$ and some $\mathbf{e} \in A^m$. Moreover,

(1) $K_p^{(m)}(a,b,c,d)$ indicates the restriction of $K_p(a,b,c,d)$ to $r \in \mathrm{Pol}_2^{(m)}(\mathbf{A})$.
(2) $K_{fg}^{(m)}(a,b,c,d)$ denotes the restriction of $K_{fg}(a,b,c,d)$ to $s,t \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$.
(3) $L_{fg}^{(m)}(a,b,c,d)$ denotes the restriction of $L_{fg}(a,b,c,d)$ to $s_1,s_2,t \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$.

Definition 5.2. Suppose $\mathbf{A}$ is an algebra, $a,b,c,d \in A$, and $m,k \geq 1$.

(1) $(a,b) \Rightarrow_{(m)} (c,d)$ iff $\{c,d\} = \{s(a), s(b)\}$ for some $s \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$.
(2) $(a,b) \Rrightarrow_{(m)}^k (c,d)$ iff there exist $c = c_0, c_1, \ldots, c_k = d$ such that $(a,b) \Rightarrow_{(m)} (c_i, c_{i+1})$ for all $i < k$.

Observe that $(a,b) \Rrightarrow_{(m)}^k (c,d)$ implies $(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$, and that $\Rrightarrow_{(m)}^k$ is first-order definable for each $m,k \geq 1$ in any locally finite variety.

Lemma 5.3. *Suppose $\mathcal{V}$ is a variety with a difference term and having residual bound m. Let $\mathbf{A} \in \mathcal{V}$ and $a,b,c,d \in A$, and put $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$.*

(1) $\vec{K}_p(a,b,c,d) \equiv \vec{K}_p^{(m)}(a,b,c,d)$.
(2) *If* $0_A \prec \alpha$, *then* $\vec{K}_{fg}(a,b,c,d) \equiv \vec{K}_{fg}^{(m)}(a,b,c,d)$.
(3) *If* $0_A \prec \alpha$, *then* $\vec{L}_{fg}(a,b,c,d)$ *iff* $\vec{L}_{fg}^{s_1,s_2,t,i}(a,b,c,d)$ *for all* $s_1,s_2 \in \mathrm{Pol}_1(\mathbf{A})$, *all* $t \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$, *and all* $i \in I$.

Proof. (1) We follow the proof of Lemma 3.5 in [19]. Assume that $\vec{K}_p^{(m)}(a,b,c,d)$ holds, and let $r \in \mathrm{Pol}_2(\mathbf{A})$. We must prove

$$p(r(a,c), r(b,c), r(b,d)) = p(r(a,d), r(b,d), r(b,d)),$$

and to do that it suffices to show that

$$p(r(a,c), r(b,c), r(b,d)) \stackrel{\theta}{\equiv} p(r(a,d), r(b,d), r(b,d)) \tag{5.1}$$

for all $\theta \in \mathrm{Con}\,\mathbf{A}$ of index at most m. Fix such θ and let T be a transversal for θ; thus $T \subseteq A$, $|T| \leq m$, and T intersects each θ-class in exactly one element. Pick a term $t(x, y, \mathbf{z})$ and parameters $\mathbf{e}$ from A so that $r(x, y) = t^{\mathbf{A}}(x, y, \mathbf{e})$. Define $\mathbf{u}$ so that u_i is the unique element of $e_i/\theta \cap T$ and define $r'(x, y) = t^{\mathbf{A}}(x, y, \mathbf{u})$. Then

- $r(x, y) \stackrel{\theta}{\equiv} r'(x, y)$ for all $x, y \in A$.
- $r' \in \mathrm{Pol}_2^{(m)}(\mathbf{A})$.

As $\vec{K}_p^{(m)}(a, b, c, d)$ holds by assumption, we have

$$p(r'(a, c), r'(b, c), r'(b, d)) = p(r'(a, d), r'(b, d), r'(b, d)),$$

which implies (5.1).

(2) We follow the main idea of the proof of Lemma 2 in [2]. Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be meet-irreducible and satisfying $\theta \cap \alpha = 0_A$. Then θ has index at most m. Assume that $\vec{K}_{fg}^{(m)}(a, b, c, d)$ holds but $\vec{K}_{fg}(a, b, c, d)$ fails at $s, t \in \mathrm{Pol}_1(\mathbf{A})$ and $i \in I$. Thus

$$\begin{array}{rcl} f_i(s(a), t(c), s(b)) & = & g_i(s(a), t(c), s(b)) \\ u := f_i(s(a), t(d), s(b)) & \neq & g_i(s(a), t(d), s(b)) =: v. \end{array}$$

Note that $(u, v) \in \alpha$, so $(u, v) \notin \theta$. As in the proof of (1), we can find $s', t' \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$ such that $s(x) \stackrel{\theta}{\equiv} s'(x)$ and $t(x) \stackrel{\theta}{\equiv} t'(x)$ for all $x \in A$. Thus

$$\begin{array}{rcl} f_i(s'(a), t'(c), s'(b)) & \stackrel{\theta}{\equiv} & g_i(s'(a), t'(c), s'(b)) \\ f_i(s'(a), t'(d), s'(b)) & \stackrel{\theta}{\not\equiv} & g_i(s'(a), t'(d), s'(b)). \end{array}$$

In addition, the difference term identities imply

$$\begin{array}{rcl} f_i(s'(a), t'(c), s'(b)) & \stackrel{\alpha}{\equiv} & f_i(s'(a), t'(c), s'(a)) \\ & = & g_i(s'(a), t'(c), s'(a)) \\ & \stackrel{\alpha}{\equiv} & g_i(s'(a), t'(c), s'(b)). \end{array}$$

As $\theta \cap \alpha = 0_A$, this proves

$$f_i(s'(a), t'(c), s'(b)) = f_i(s'(a), t'(c), s'(a))$$

which contradicts $\vec{K}_{fg}^{(m)}(a, b, c, d)$.

(3) The proof is similar to the proof of item (2). □

Theorem 5.4. *Suppose $\mathcal{V}$ is a variety with a difference term and having a finite residual bound m. $C(x, y, z, w)$ is equivalent in $\mathcal{V}$ to the following condition:*

(∗) *For all x_1, y_1, z_1, w_1, if $(x, y) \Rrightarrow^2_{(m+3)} (x_1, y_1)$ and $(z, w) \Rrightarrow^2_{(m+3)} (z_1, w_1)$, then $K_p^{(m)}(x_1, y_1, z_1, w_1)$ & $K_p^{(m)}(z_1, w_1, x_1, y_1)$ & $K_{fg}^{(m)}(z_1, w_1, x_1, y_1)$.*

Proof. Suppose $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$. Clearly $C(a, b, c, d)$ implies the above condition. For the remainder of the proof, assume (∗) holds and yet $\neg C(a, b, c, d)$; we will find a contradiction. We may assume that $\mathbf{A}$ is finite.

Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$ and $\beta = \mathrm{Cg}^{\mathbf{A}}(c,d)$. By condition $(*)$ and Lemma 5.3(1), we have $K_p(a,b,c,d)$ and $K_p(c,d,a,b)$. Hence by Corollary 4.10 and Lemma 4.5(3), $\alpha\cap\beta$ is not abelian. Choose $\gamma \in \mathrm{Con}\,\mathbf{A}$ with $0_A \prec \gamma \leq [\alpha\cap\beta, \alpha\cap\beta]$. Let θ be a maximal congruence satisfying $\gamma \not\leq \theta$, and let μ be the unique upper cover of θ. Also let $\nu = (0_A : \gamma)$, so $\nu = (\theta : \mu)$ by Lemma 2.5. Observe that $\gamma \cap \theta = 0_A$ implies $\theta \leq \nu$, which with $C(\nu, \mu; \theta)$ implies $[\nu/\theta, \mu/\theta] = 0_{A/\theta}$.

Because $\mathcal{V}$ has a difference term and is residually small, it satisfies C1 by Proposition 3.1(2). Applied to $\mathbf{A}/\theta$ and the previous commutator fact, this gives $\mu/\theta \not\leq [\nu/\theta, \nu/\theta]$, so $[\nu/\theta, \nu/\theta] = 0_{A/\theta}$, which implies $[\nu,\nu] \leq \theta$.

Observe that if α centralized γ, then we would have $\alpha \leq \nu$ and hence

$$\gamma \leq [\alpha,\alpha] \leq [\nu,\nu] \leq \theta,$$

which is false. This proves that α does not centralize ν. Similarly, β does not centralize ν.

Pick $(u,v) \in \gamma \setminus 0_A$. By what we have just proved and symmetry of the centralizer relation, we have $\neg C(u,v,a,b)$. Thus by Corollary 4.10, at least one of $K_p(u,v,a,b)$ or $L_{fg}(u,v,a,b)$ must fail. Suppose first that $K_p(u,v,a,b)$ fails; then $K_p^{(m)}(u,v,a,b)$ fails by Lemma 5.3(1). Pick $r \in \mathrm{Pol}_2^{(m)}(\mathbf{A})$ witnessing the failure; thus

$$a_1 := p(r(u,a), r(v,a), r(v,b)) \neq p(r(u,a), (v,b), (v,b)) =: b_1.$$

Note that $(a_1,b_1) \in \gamma \setminus 0_A$ and $(a,b) \Rightarrow_{(m+3)} (a_1,b_1)$ witnessed by the polynomial $s(x) = p(u', r(v,x), v')$ where $u' = r(u,a)$ and $v' = r(v,b)$. Suppose instead that $L_{fg}(u,v,a,b)$ fails; pick $s_1, s_2, t \in \mathrm{Pol}_1(\mathbf{A})$ and $i \in I$ such that, without loss of generality,

$$\begin{aligned} s_1(u) &= s_2(u) \\ f_i(s_1(v), t(a), s_2(v)) &= g_i(s_1(v), t(a), s_2(v)) \\ a_1 := f_i(s_1(v), t(b), s_2(v)) &\neq g_i(s_1(v), t(b), s_2(v)) =: b_1. \end{aligned}$$

By Lemma 5.3(3), we may assume that $t \in \mathrm{Pol}_1^{(m)}(\mathbf{A})$. Note that $(a_1,b_1) \in \gamma \setminus 0_A$ and $(a,b) \Rrightarrow^2_{(m+2)} (a_1,b_1)$ witnessed by the polynomials $f_i(u', t(x), v')$ and $g_i(u', t(x), v')$ where $u' = s_1(v)$ and $v' = s_2(v)$.

Thus in either case, we have established the existence of $(a_1,b_1) \in \gamma \setminus 0_A$ with $(a,b) \Rrightarrow^2_{(m+3)} (a_1,b_1)$. A similar argument proves the existence of $(c_1,d_1) \in \gamma \setminus 0_A$ with $(c,d) \Rrightarrow^2_{(m+3)} (c_1,d_1)$. Choose and fix such a_1,b_1,c_1,d_1.

By condition $(*)$, we have both $K_p^{(m)}(a_1,b_1,c,d)$ and $K_p^{(m)}(c,d,a_1,b_1)$. Hence by Lemma 5.3(1) we have $K_p(a_1,b_1,c,d)$ and $K_p(c,d,a_1,b_1)$. As $C(c,d,a_1,b_1)$ fails (because β does not centralize γ), Lemma 4.5(3) and Corollary 4.10 imply that $\beta \cap \gamma = \gamma$ is nonabelian. Let U be a $(0,\gamma)$-minimal set with trace $\{0,1\}$. Let $r \in \mathrm{Pol}_2(\mathbf{A})$ be a polynomial whose range is U and whose restriction to $\{0,1\}$ is the meet semilattice operation. As $(a_1,b_1), (c_1,d_1) \in \gamma \setminus 0_A$, there exist $s,t \in \mathrm{Pol}_1(\mathbf{A})$

such that $\{s(a_1), s(b_1)\} = \{t(c_1), t(d_1)\} = \{0, 1\}$. Define $r'(x, y) = r(s(x), t(y))$. Then three of $r'(a_1, c_1), r'(a_1, d_1), r'(b_1, c_1), r'(b_1, d_1)$ equal 0 while the fourth equals 1. Hence $H(a_1, b_1, c_1, d_1)$ fails at r'. By Lemma 4.9(2), one of $K_p(c_1, d_1, a_1, b_1)$ or $K_{fg}(c_1, d_1, a_1, b_1)$ must fail. Lemma 5.3 then implies that one of $K_p^{(m)}(c_1, d_1, a_1, b_1)$ or $K_{fg}^{(m)}(c_1, d_1, a_1, b_1)$ must fail, contradicting condition $(*)$. $\square$

6. The Kiss 4-ary term

Throughout this section, $\mathcal{V}$ is a variety having a difference term p. Define the associated Kiss 4-ary term by Lipparini's Formula $q(x, y, z, w) := p(p(x, z, z), p(y, w, z), z)$.[2] Following Kiss we call (a, b, c, d) an α, β-rectangle if $(a, b), (c, d) \in \alpha$ and $(a, c), (b, d) \in \beta$, and we let $R(\alpha, \beta)$ be the set of these. $R(\alpha, \beta)$ is a subuniverse of $\mathbf{A}^4$.

Lemma 6.1 ([18])**.** *If $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con}(\mathbf{A})$, then*

(1) $\mathcal{V} \models q(x, y, x, y) \approx x$,
(2) $\mathcal{V} \models q(x, x, y, y) \approx y$, *and*
(3) $q(a, b, c, d) \equiv_{[\beta,\alpha]} q(a, b, c', d)$ *if* $(a, b, c, d), (a, b, c', d) \in R(\alpha, \beta)$. $\square$

Lemma 6.2. *If $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con}(\mathbf{A})$, then $[\alpha, \beta] = 0$ iff*

(i) *$q\colon R(\alpha, \beta) \to \mathbf{A}$ is a homomorphism, and*
(ii) *q is independent of its third variable on $R(\alpha, \beta)$.*

Proof. In the case where $\mathcal{V}$ is congruence modular this lemma is Theorem 3.8 (iii) of [14]. The proof below follows the argument from page 472 of [14].

Let $\Delta_{\alpha,\beta}$ be the congruence on $\mathbf{A} \times_\alpha \mathbf{A}$ generated by the β-diagonal. Kiss argues that if $(a_i, b_i, c_i, d_i) \in R(\alpha, \beta)$, then for any term s we have

(6.1)
$$(q(s(\mathbf{a}), s(\mathbf{b}), s(\mathbf{c}), s(\mathbf{d})), s(\mathbf{d})) \equiv_{\Delta_{\alpha,\beta}} (s(\mathbf{a}), s(\mathbf{b})) \equiv_{\Delta_{\alpha,\beta}} (s(\overline{q(a_i, b_i, c_i, d_i)}), s(\mathbf{d})).$$

The argument he gives works under our hypotheses. Kiss then uses a property of the modular commutator to derive from (6.1) that

$$q(s(\mathbf{a}), s(\mathbf{b}), s(\mathbf{c}), s(\mathbf{d})) \equiv_{[\alpha,\beta]} s(\overline{q(a_i, b_i, c_i, d_i)}).$$

A justification that this step works under our hypotheses is required.

If $\mathcal{V}$ has a difference term, then it satisfies a nontrivial idempotent Maltsev condition. Lemma 4.4 of [13] shows (with a slight change of notation) that if $[\alpha, \beta] = 0$, then on $\mathbf{A} \times_\alpha \mathbf{A}$ it is the case that

$$\beta_1 \wedge 0_2 \wedge \Delta_\alpha = 0.$$

Here β_1 is the congruence on $\mathbf{A} \times_\alpha \mathbf{A}$ that relates pairs whose first coordinates are β-related, 0_2 is the congruence on $\mathbf{A} \times_\alpha \mathbf{A}$ that relates pairs whose second coordinates

[2]Lipparini's difference term has its variables in the reverse order of Kiss's difference term. Kiss's convention agrees with ours, so this formula looks different from the one in Lipparini's paper.

are equal, and Δ_α is the largest congruence on $\mathbf{A}\times_\alpha\mathbf{A}$ which relates no diagonal pair to any off-diagonal pair.

The two sides of (6.1) are equal in the second coordinate, hence are 0_2-related. Since $\Delta_{\alpha,\beta}\subseteq\Delta_\alpha$, as a consequence of $[\beta,\alpha]=[\alpha,\beta]=0$, we get that the two sides of (6.1) are Δ_α-related. Since

$$q(s(\mathbf{a}),s(\mathbf{b}),s(\mathbf{c}),s(\mathbf{d}))\equiv_\beta q(s(\mathbf{a}),s(\mathbf{b}),s(\mathbf{a}),s(\mathbf{b}))=s(\mathbf{a}),$$

and similarly $s(\overline{q(a_i,b_i,c_i,d_i)})\equiv_\beta s(\overline{q(a_i,b_i,a_i,b_i)})=s(\mathbf{a})$, we get that the two sides of (6.1) are β_1-related. Altogether we get the desired conclusion, that

$$q(s(\mathbf{a}),s(\mathbf{b}),s(\mathbf{c}),s(\mathbf{d}))=s(\overline{q(a_i,b_i,c_i,d_i)})$$

when $[\alpha,\beta]=0$. This is the property that $q\colon R(\alpha,\beta)\to\mathbf{A}$ is a homomorphism, hence item (i) holds if $[\alpha,\beta]=[\beta,\alpha]=0$. We get that item (ii) also holds from Lemma 6.1 (3).

Now we prove that (i) and (ii) force $[\beta,\alpha]=0$. Define

$$\Delta=\{((a,b),(q(a,b,c,d),d))\mid(a,b,c,d)\in R(\alpha,\beta)\}.$$

Kiss shows that Δ is a congruence on $\mathbf{A}\times_\alpha\mathbf{A}$ that contains $\Delta_{\alpha,\beta}$. If the first pair in the pair of pairs, $((a,b),(q(a,b,c,d),d))\in\Delta$, lies on the diagonal (a complicated way of writing "if $a=b$"), then

$$q(a,b,c,d)=q(a,a,c,d)=q(a,a,d,d)=d, \tag{6.2}$$

and the second pair in the pair of pairs also lies on the diagonal. (In the middle equality of (6.2) we are using that q is independent of its third variable on α,β-rectangles.) Since $\Delta_{\alpha,\beta}\subseteq\Delta$, and Δ relates no diagonal pair of $\mathbf{A}\times_\alpha\mathbf{A}$ to an off-diagonal pair, we derive that $[\beta,\alpha]=0$ holds. $\square$

7. The finite basis argument

In this final section we prove Theorem 1.3. Our strategy is to mimic McKenzie's argument [19, Section 4] for the congruence modular case, to the extent that that is possible. Parenthetical references are to the corresponding results from [19]. Some technical issues in McKenzie's argument become easier here because of our use of the Kiss term. We are forced to give an entirely new proof of the final step in establishing C1 (i.e., property (4) of Proposition 3.3).

Let $\mathcal{V}_0$ be a finitely based variety in a finite language $\mathcal{L}$ with a difference term, and let $\mathcal{V}$ be a subvariety of $\mathcal{V}_0$ with a finite residual bound r. For each $j\geq 3$ let $\mathcal{V}^{(j)}$ be the subvariety of $\mathcal{V}_0$ axiomatized by the j-variable identities of $\mathcal{V}$. Let $X=(\mathsf{x}_1,\mathsf{x}_2,\ldots)$ be a fixed infinite sequence of variables. Define the *height* of a term in some standard way, so that for each $n,h\geq 0$, the set $\mathrm{Trm}_n^{(h)}(\mathcal{L})$ of $\mathcal{L}$-terms over $\{\mathsf{x}_1,\ldots,\mathsf{x}_n\}$ of height at most h is a finite set closed under subterms. Let $h_\mathcal{V}:\omega\to\omega$ be a function so that for all $n\geq 0$, every $\mathcal{L}$-term over $\{\mathsf{x}_1,\ldots,\mathsf{x}_n\}$ is equivalent modulo

$\mathcal{V}$ to a term in $\mathrm{Trm}_n^{(h_{\mathcal{V}}(n))}(\mathcal{L})$. For simplicity, we denote $\mathrm{Trm}_n^{(h_{\mathcal{V}}(n))}(\mathcal{L})$ by $\mathrm{Trm}_n(\mathcal{V})$. For each $n > 0$ let σ_n be a sentence asserting $\forall \mathbf{x}[f(s_1(\mathbf{x}), \ldots, s_k(\mathbf{x})) = t(\mathbf{x})]$ for all k-ary fundamental operation symbols of $\mathcal{L}$ and all $s_1, \ldots, s_k, t \in \mathrm{Trm}_n(\mathcal{V})$ such that $\mathcal{V} \models f(s_1, \ldots, s_k) \approx t$. Thus σ_j is a finite axiomatization of $\mathcal{V}^{(j)}$ relative to $\mathcal{V}_0$.

Lemma 7.1 (Lemma 4.1)**.** *There exists a first-order formula $\Omega(x, y, z, w)$ such that:*

(1) $\mathcal{V} \models C(x, y, z, w) \leftrightarrow \Omega(x, y, z, w)$.
(2) $\mathcal{V}_0 \models C(x, y, z, w) \to \Omega(x, y, z, w)$.
(3) *There exists $m > 0$ such that for all sufficiently large j,*

$$\mathcal{V}^{(j)} \models \Omega(x, y, z, w) \leftrightarrow [K_p^{(m)}(x, y, z, w) \ \&\ L_{fg}^{(m)}(x, y, z, w)].$$

(4) *There exists an existential first-order formula $W(u, v, x, y, z, w)$ satisfying*
  (a) $\mathcal{V}_0 \models W(u, v, x, y, z, w) \to$ "$(u, v) \in [\mathrm{Cg}(x, y), \mathrm{Cg}(z, w)]$".
  (b) *For all sufficiently large j,*

$$\mathcal{V}^{(j)} \models \Omega(x, y, z, w) \leftrightarrow \forall u, v[W(u, v, x, y, z, w) \to u = v].$$

Proof. Start with the condition $(*)$ expressed in Theorem 5.4 (with m replaced by r). Modulo σ_{r+4}, $(x, y) \Rrightarrow^3_{(r+3)} (u, v)$ is equivalent to its restriction to unary polynomials defined from terms in $\mathrm{Trm}_{r+4}(\mathcal{V})$. Similarly, modulo σ_{r+2}, $K_p^{(r)}(x, y, z, w)$ is equivalent to its restriction to binary polynomials defined from terms in $\mathrm{Trm}_{r+2}(\mathcal{V})$, and modulo σ_{r+1}, $K_{fg}^{(r)}(x, y, z, w)$ is equivalent to its restriction to unary polynomials defined from terms in $\mathrm{Trm}_{r+1}(\mathcal{V})$. Hence in models of σ_{r+1} & σ_{r+2} & σ_{r+4} (in particular, in $\mathcal{V}$), the condition $(*)$ can be expressed by a first-order formula $\Omega(x, y, z, w)$. This proves (1). As $\Omega(x, y, z, w)$ is a special case of $(*)$, which in turn is implied by $C(x, y, z, w)$ in $\mathcal{V}_0$, we get (2).

For $m, h > 0$ let $K_p^{(m,h)}(x, y, z, w)$ denote the restriction of $K_p^{(m)}(x, y, z, w)$ to binary polynomials definable from terms in $\mathrm{Trm}_{m+2}^{(h)}(\mathcal{L})$, and let $L_{fg}^{(m,h)}(x, y, z, w)$ be the restriction of $L_{fg}^{(m)}(x, y, z, w)$ to unary polynomials definable from terms in $\mathrm{Trm}_{m+1}^{(h)}(\mathcal{L})$. Because

$$(\dagger) \qquad \mathcal{V}_0 \models C(x, y, z, w) \leftrightarrow \bigwedge_{m,h>0} (K_p^{(m,h)}(x, y, z, w) \ \&\ L_{fg}^{(m,h)}(x, y, z, w))$$

by Corollary 4.10, and because $K_p^{(m,h)}(x, y, z, w)$ and $L_{fg}^{(m,h)}(x, y, z, w)$ are expressible by a first-order formulas for each fixed $m, h > 0$, the compactness theorem with (2) and $(\dagger)$ imply the existence of $m, h > 0$ such that $\mathcal{V}_0 \models (K_p^{(m,h)} \ \&\ L_{fg}^{(m,h)}) \to \Omega$. Thus $\mathcal{V} \models \Omega \leftrightarrow (K_p^{(m,h)} \ \&\ L_{fg}^{(m,h)})$, so again by the compactness theorem,

$$(\ddagger) \qquad \mathcal{V}^{(j)} \models \Omega \leftrightarrow (K_p^{(m,h)} \ \&\ L_{fg}^{(m,h)}) \quad \text{for all sufficiently large } j.$$

We may assume $h \geq h_{\mathcal{V}}(m+2)$. By the compactness theorem, for all sufficiently large j, $\mathcal{V}^{(j)}$ models σ_{m+1} & σ_{m+2} and hence satisfies $K_p^{(m,h)} \equiv K_p^{(m)}$ and $L_{fg}^{(m,h)} \equiv L_{fg}^{(m)}$. This and (‡) prove (3).

Recall that $K_p^{(m,h)}(x,y,z,w)$ is a conjunction of equations, while $L_{fg}^{(m,h)}(x,y,z,w)$ is a conjunction of quasi-equations. Given $\mathbf{A} \in \mathcal{V}_0$ and $a,b,c,d,u,v \in A$, call (u,v) an (m,h)-*critical pair for* (a,b,c,d) if there exists an equation in $K_p^{(m,h)}(a,b,c,d)$ whose left and right sides are u,v respectively, or there exists a quasi-equation in $L_{fg}^{(m,h)}(a,b,c,d)$ whose conclusion is the equation with left and right sides u,v respectively. We can take $W(u,v,x,y,z,w)$ to be a first-order sentence which asserts that (u,v) is an (m,h)-critical pair for (x,y,z,w). (4a) is then obvious, and (4b) follows from (‡). □

Definition 7.2. Given $\mathbf{A} \in \mathcal{V}_0$ and $a,b \in A$, let

$$\begin{aligned}\Omega(a,b) &:= \{(x,y) : \Omega(x,y,a,b)\}\\ \Omega_{\mathrm{op}}(a,b) &:= \{(x,y) : \Omega(x,y,z,w) \text{ for all } (z,w) \in \Omega(a,b)\}\end{aligned}$$

Lemma 7.3 (Lemma 4.4)**.** *For all sufficiently large j, all $\mathbf{A} \in \mathcal{V}^{(j)}$, and all $a,b \in A$, $\Omega(a,b)$ and $\Omega_{\mathrm{op}}(a,b)$ are congruences.*

Proof. In $\mathcal{V}$, $\Omega(a,b) = \mathrm{ann}(a,b)$ and $\Omega_{\mathrm{op}}(a,b) = (0_A : \mathrm{ann}(a,b))$. Hence the claim is true in $\mathcal{V}$, and as it can be expressed by a first-order sentence, is true in $\mathcal{V}^{(j)}$ for all sufficiently large j by the compactness theorem. □

Lemma 7.4. *For all sufficiently large j, $\mathcal{V}^{(j)} \models C(x,y,z,w) \leftrightarrow \Omega(x,y,z,w)$.*

Proof. By Lemma 7.1, $C(x,y,u,v) \to \Omega(x,y,z,w)$ holds in $\mathcal{V}_0$.

By Lemma 7.3, the relations $\Omega(a,b)$ and $\Omega_{\mathrm{op}}(a,b)$ are congruences for any (a,b) in $\mathcal{V}^{(j)}$ for j sufficiently large, and their definitions yield that $\Omega(x,y,z,w)$ holds for any $(x,y) \in \Omega(a,b)$ and any $(z,w) \in \Omega_{\mathrm{op}}(a,b)$. We can write a first-order sentence that asserts (in an algebra $\mathbf{A}$) that for all $a,b \in A$, (i) $q\colon R(\Omega(a,b),\Omega_{\mathrm{op}}(a,b)) \to \mathbf{A}$ is a homomorphism and (ii) q is independent of its third variable on $R(\Omega(a,b),\Omega_{\mathrm{op}}(a,b))$. This sentence is true in $\mathcal{V}$ by Lemma 7.1(1) and Lemma 6.2, so is true in $\mathcal{V}^{(j)}$ for sufficiently large j. Hence by Lemma 6.2, $[\Omega(z,w),\Omega_{\mathrm{op}}(z,w)] = 0$ holds in $\mathcal{V}^{(j)}$ for sufficiently large j. But then in $\mathcal{V}^{(j)}$, we must have $\Omega(x,y,z,w) \to C(x,y,z,w)$, because if $\Omega(x,y,z,w)$ holds, then $(x,y) \in \Omega(z,w)$, while $(z,w) \in \Omega_{\mathrm{op}}(z,w)$ always holds. □

Definition 7.5. Let $\mu(x,y)$ be the formula $\Omega(x,y,x,y)$.

Corollary 7.6 (Lemma 4.5)**.** *For all sufficiently large j, $\mathbf{A} \in \mathcal{V}^{(j)}$, and $a,b,c \in A$,*

(1) $\mathbf{A} \models \mu(a,b)$ *iff* $\mathrm{Cg}^{\mathbf{A}}(a,b)$ *is abelian.*
(2) $\Omega(a,b) = \mathrm{ann}(a,b)$.
(3) $\Omega_{\mathrm{op}}(a,b) = (0_A : \mathrm{ann}(a,b))$.

Proof. Follows easily from Lemma 7.4. □

Lemma 7.7 (Lemma 4.9)**.** *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$ and $a, b, e_0, \ldots, e_r \in A$ with $a \neq b$, then there exists (c, d) satisfying:*

1. *$(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b) \setminus 0_A$.*
2. *$C(c, d, e_i, e_j)$ for some $0 \leq i < j \leq r$.*

Proof sketch. The argument is a little different than in the congruence modular case as we haven't established [19, Lemma 4.8].

Assume j is large enough to satisfy the claims in Lemmas 7.1, 7.3, 7.4 and Corollary 7.6. Let $(u_0, v_0) = (a, b)$. If $C(u_0, v_0, e_0, e_1)$, then we're done. Otherwise, we have $\neg\Omega(u_0, v_0, e_0, e_1)$, and by Lemma 7.1(4) this is witnessed by a critical pair (u_1, v_1) satisfying $u_1 \neq v_1$ and $W(u_1, v_1, u_0, v_0, e_0, e_1)$. This implies that $(u_1, v_1) \in [\mathrm{Cg}(u_0, v_0), \mathrm{Cg}(e_0, e_1)]$ and hence that $(u_1, v_1) \in \mathrm{Cg}(a, b) \cap \mathrm{Cg}(e_0, e_1)$. Next, check whether $C(u_1, v_1, e_0, e_2)$; again if true we're done, while if false then the failure gives a critical pair (u_2, v_2) with $u_2 \neq v_2$ and $W(u_2, v_2, u_1, v_1, e_0, e_2)$. This again implies $(u_2, v_2) \in \mathrm{Cg}(u_1, v_1) \cap \mathrm{Cg}(e_0, e_2)$. We can proceed in this way through all $M := \binom{r+1}{2}$ pairs (e_i, e_j). As r is fixed, if we never find what we want, we end up with a system of short proofs of $(u_{t+1}, v_{t+1}) \in \mathrm{Cg}(u_t, v_t)$ for $1 \leq t \leq M$, so that $u_t \neq v_t$ for all t, and for all $0 \leq i < j \leq r$ there exists t and a short proof of $(u_t, v_t) \in \mathrm{Cg}(e_i, e_j)$. This is a first-order definable configuration. Any algebra in which it occurs has a subdirectly irreducible quotient of cardinality greater than r. This cannot occur in $\mathcal{V}$, so by the compactness theorem, it cannot occur in $\mathcal{V}^{(j)}$ for sufficiently large j. □

Lemma 7.8 (Lemma 4.10)**.** *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$ is finitely generated and $a, b \in A$ with $a \neq b$, then there exists $(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b) \setminus 0_A$ such that $|A/\operatorname{ann}(c, d)| \leq r$.*

Proof. Identical to the proof of [19, Lemma 4.10]. □

Corollary 7.9 (Lemma 4.13; cf. Lemma 4.19)**.** *For all sufficiently large j:*

1. *If $\mathbf{A} \in \mathcal{V}^{(j)}$ and $0_A \prec \alpha \in \mathrm{Con}\,\mathbf{A}$, then $|A/(0_A : \alpha)| \leq r$.*
2. *There exist first-order formulas $\mathrm{AbAt}(x, y)$ and $\theta(u, v, x, y)$, not depending on j, such that for all $\mathbf{A} \in \mathcal{V}^{(j)}$ and all $c, d \in A$, letting $\alpha = \mathrm{Cg}^{\mathbf{A}}(c, d)$:*
   (a) *$\mathbf{A} \models \mathrm{AbAt}(c, d)$ iff α is an abelian atom in $\mathrm{Con}\,\mathbf{A}$.*
   (b) *if α is an abelian atom, then $\alpha = \{(a, b) \in A^2 : \mathbf{A} \models \theta(a, b, c, d)\}$.*

Proof. (1) Suppose $0_A \prec \alpha = \mathrm{Cg}^{\mathbf{A}}(a, b)$. Let $\gamma = \operatorname{ann}(a, b)$ and suppose $|A/\gamma| > r$. Pick $e_0, \ldots, e_r \in A$ so that no two are related by γ. By Lemma 7.7 there exists $(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$ with $c \neq d$ and $C(c, d, e_i, e_j)$ for some $i < j$. But then $\mathrm{Cg}^{\mathbf{A}}(c, d) = \alpha$ and $(e_i, e_j) \in \operatorname{ann}(c, d) = \gamma$, a contradiction. Thus $|A/\gamma| \leq r$, which proves (1).

(2) Let $\theta(u, v, x, y)$ be the following formula:

$$\theta(u, v, x, y) \quad : \quad \bigvee_{t \in \mathrm{Trm}_{r+1}(\mathcal{V})} \exists e_1 \cdots e_r [p(t(x, \mathbf{e}), t(y, \mathbf{e}), u) = v].$$

We can assume $\mathcal{V}^{(j)} \models \sigma_{r+1}$; hence if $\mathbf{A} \in \mathcal{V}^{(j)}$ and $c, d \in A$, then the set $\{(a,b) : \mathbf{A} \models \theta(a,b,c,d)\}$ coincides with $\Gamma_r(c,d)$ from Definition 2.6. Thus (2b) follows from Lemma 2.7(1). Now let $\mathrm{AbAt}(x,y)$ be a formula expressing the following:

$$x \neq y \ \&\ \mu(x,y) \ \&\ \text{"}\{(a,b) : \theta(a,b,x,y)\} \text{ is a congruence containing } (x,y)\text{"}$$
$$\&\ \forall u,v[(\theta(u,v,x,y) \ \&\ u \neq v) \ \rightarrow\ \theta(x,y,u,v)].$$

That $\mathrm{AbAt}(x,y)$ has the claimed property follows from (1), Corollary 7.6, Lemma 2.7, and the fact that $\Gamma_r(c,d) \subseteq \mathrm{Cg}^{\mathbf{B}}(c,d)$ for any $\mathbf{B} \in \mathcal{V}_0$ and $c, d \in B$. □

Lemma 7.10 (Lemma 4.15)**.** *For all sufficiently large j, $\mathcal{V}^{(j)}$ is locally finite.*

Proof. If not, we can find a finitely generated infinite algebra $\mathbf{A} \in \mathcal{V}^{(j)}$ such that every nonzero congruence of $\mathbf{A}$ has finite index. Using Lemma 7.8, there exists a nonzero congruence β such that $(0_A : \beta)$ has finite index. Then $\beta \cap (0_A : \beta)$ also has finite index so is nonzero. As $\beta \cap (0_A : \beta)$ is abelian, this proves the existence of a nonzero *abelian* congruence α. Using Lemma 7.8 again, we get $(a,b) \in \alpha \setminus 0_A$ with $\mathrm{ann}(a,b)$ having index at most r. But Lemma 2.7(2) then says that each $\mathrm{Cg}^{\mathbf{B}}(a,b)$-block is finite, which is impossible. □

Next, we work towards establishing that $\mathcal{V}^{(j)}$ satisfies the commutator identity C1. (Recall that $\mathcal{V}$ itself satisfies C1 by Proposition 3.1(2).) Our strategy will be to verify each of the conditions in Proposition 3.3.

Definition 7.11. Given $\mathbf{A} \in \mathcal{V}_0$, let

$$\mu^{\mathbf{A}} := \{(x,y) \in A^2 : \mu(x,y)\}$$

where $\mu(x,y)$ is the formula from Definition 7.5.

Lemma 7.12 (Lemma 4.17(1))**.** *For all sufficiently large j and all $\mathbf{A} \in \mathcal{V}^{(j)}$, $\mu^{\mathbf{A}}$ is the largest abelian congruence of $\mathbf{A}$.*

Proof. As $\mu^{\mathbf{A}}$ contains every abelian congruence by Corollary 7.6(1), it suffices to prove that $\mu^{\mathbf{A}}$ is itself abelian. This property is first-order by Lemma 6.2, so it suffices to prove this latter claim for $\mathbf{A} \in \mathcal{V}$. Fix $\mathbf{A} \in \mathcal{V}$ and suppose α, β are abelian congruences. Let $\gamma = \alpha \vee \beta$ and $\delta = [\alpha, \beta]$. We have $\delta \leq [\gamma, \gamma]$ by monotonicity, so $[\gamma, \delta] = \delta$ by C1. On the other hand, $[\alpha, \delta] \leq [\alpha, \alpha] = 0_A$ so $\alpha \leq (0_A : \delta)$, and similarly $\beta \leq (0_A : \delta)$. Thus $\gamma \leq (0_A : \delta)$, which means $[\gamma, \delta] = 0_A$. This proves $[\alpha, \beta] = 0_A$ whenever α, β are abelian congruences. Now use [10, Lemma 2.8] to deduce that the join of all abelian congruences of $\mathbf{A}$ is itself abelian; call it $\alpha_{\max}$. Hence

$$\mu^{\mathbf{A}} \ = \bigvee_{(a,b) \in \mu^{\mathbf{A}}} \mathrm{Cg}^{\mathbf{A}}(a,b) \ \subseteq \bigvee_{\alpha \text{ abelian}} \alpha \ = \ \alpha_{\max}$$

which proves $\mu^{\mathbf{A}}$ is abelian. □

Recall that m is fixed satisfying Lemma 7.1(3). The next definition simply gives notation for the set of critical pairs for $K_p^{(6m+1)}(a,b,a,b)$.

Definition 7.13. For $\mathbf{A} \in \mathcal{V}_0$ and $a, b \in A$, define

$$\begin{aligned} G_m(a,b) \;\; = \;\; \{\, &(p(r(x,z), r(y,z), r(y,w)),\, p(r(x,w), r(y,w), r(y,w))) : \\ & r \in \mathrm{Pol}_2^{(6m+1)}(\mathbf{A}),\ \{x,y\} = \{z,w\} = \{a,b\} \,\}. \end{aligned}$$

Definition 7.14. Let $\mathbf{A} \in \mathcal{V}_0$ and $u, v \in A$. We say that (u,v) is a *p-snag* if $u \neq v$ and $p(p(u,v,v),v,v) = v$.

Lemma 7.15. *Suppose $\mathbf{A} \in \mathcal{V}_0$, (u,v) is a p-snag, and $\gamma = \mathrm{Cg}^{\mathbf{A}}(u,v)$.*

(1) $[\gamma,\gamma] = \gamma$.
(2) *(u,v) is not contained in any solvable congruence of $\mathbf{A}$.*

Proof. Let $\delta = [\gamma,\gamma]$. Then in $\mathbf{A}/\delta$ we have $\overline{\gamma} := \gamma/\delta$ is abelian, so p is Maltsev on $\overline{\gamma}$-blocks. Thus $\overline{u} = p(p(\overline{u},\overline{v},\overline{v}),\overline{v},\overline{v}) = \overline{v}$, implying $(u,v) \in \delta$, so $\delta = \gamma$. This proves (1), which obviously implies (2). □

Lemma 7.16. *For all sufficiently large j:*

(1) *If $\mathbf{A} \in \mathcal{V}^{(j)}$, $a, b \in A$, $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$ is not abelian, and $K_p^{(6m+1)}(a,b,a,b)$, then there exists a p-snag in $[\alpha,\alpha]$.*
(2) *For all sufficiently large j, $\mathcal{V}^{(j)} \models G_m(a,b) \subseteq \Omega(a,b) \to K_p^{(6m+1)}(a,b,a,b)$.*

Proof. (1) If α is not abelian then we have $\neg\Omega(a,b,a,b)$ by Lemma 7.4. If in addition $K_p^{(6m+1)}(a,b,a,b)$, then $\neg L_{fg}^{(m)}(a,b,a,b)$ by Lemma 7.1(3). The proof of Lemma 4.8(1) then gives a failure of $H_2(a,b,a,b)$ at some $r_1, r_2 \in \mathrm{Pol}_2^{(3m)}(\mathbf{A})$. $K_p^{(6m+1)}(a,b,a,b)$ and Lemma 4.5(2) then give a p-snag in $[\alpha,\alpha]$.

(2) Arguing as in the proof of Lemma 7.1(4), we can assume that $\mathcal{V}^{(j)} \models K_p^{(6m+1)} \leftrightarrow K_p^{(6m+1,h_{\mathcal{V}}(6m+3))}$. By the same device, the set $G_m(a,b)$ can be defined (uniformly in $\mathbf{A} \in \mathcal{V}^{(j)}$ and $a, b \in A$, for sufficiently large j) by a first-order formula. Hence the claim to be established can be expressed by a first-order sentence, so it suffices by the compactness theorem to prove that it holds for $\mathbf{A} \in \mathcal{V}$. Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$ and $\delta = \mathrm{Cg}^{\mathbf{A}}(G_m(a,b))$, and observe that $\Omega(a,b) = \mathrm{ann}(a,b)$ by Corollary 7.6(2). It should be clear that $\delta \leq [\alpha,\alpha]$, so by C1 we have $[\alpha,\delta] = \delta$. However the hypothesis implies $[\alpha,\delta] = 0_A$, so $\delta = 0_A$, implying $G_m(a,b) \subseteq 0_A$, which means $K_p^{(6m+1)}(a,b,a,b)$. □

Lemma 7.17 (Lemma 4.7(2))**.** *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$ and $\beta \in \mathrm{Con}\,\mathbf{A}$, then $[\beta,[\beta,\beta]] = [\beta,\beta]$.*

Proof. We first show that if $a, b \in A$, $\alpha = \mathrm{Cg}^{\mathbf{A}}(a,b)$, and $[\alpha,[\alpha,\alpha]] = 0_A$, then $[\alpha,\alpha] = 0_A$. Observe that the hypothesis implies $G_m(a,b) \subseteq \Omega(a,b)$, so $K_p^{(6m+1)}(a,b,a,b)$ by Lemma 7.16(2). The hypothesis also implies that α is solvable, so α is abelian by Lemmas 7.15 and 7.16(1), giving $[\alpha,\alpha] = 0_A$ as claimed.

Now suppose $\beta \in \operatorname{Con} \mathbf{A}$ and $[\beta,[\beta,\beta]] < [\beta,\beta]$. Let $\delta = [\beta,[\beta,\beta]]$. For $\theta \in \operatorname{Con} \mathbf{A}$ satisfying $\theta \geq \delta$ let $\overline{\theta} = \theta/\delta \in \operatorname{Con} \mathbf{A}/\delta$. Then by Lemma 2.4(2), $[\overline{\beta},\overline{\beta}] = \overline{[\beta,\beta]} > 0_{A/\delta}$ and $[\overline{\beta},[\overline{\beta},\overline{\beta}]] = \overline{[\beta,[\beta,\beta]]} = 0_{A/\delta}$. Thus by passing to $\mathbf{A}/\delta$ we can assume that $\delta = 0_A$. Observe next that $[\beta,\beta] \neq 0_A$ implies $\beta \not\leq \mu^{\mathbf{A}}$; pick $(a,b) \in \beta \setminus \mu^{\mathbf{A}}$ and put $\alpha = \operatorname{Cg}^{\mathbf{A}}(a,b)$. Then we still have $[\alpha,\alpha] > 0_A$ but $[\alpha,[\alpha,\alpha]] \leq [\beta,[\beta,\beta]] = 0_A$, contradicting the previous paragraph. □

Lemma 7.18 (cf. the proof of Lemma 4.20)**.** *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$ and $\alpha_0,\alpha_1,\beta_1,\beta_2 \in \operatorname{Con} \mathbf{A}$ with β_1,β_2 principal, $0_A \prec \alpha_0 \prec \alpha_1$, α_1 abelian, and $[\alpha_0,\beta_1] = [\alpha_0,\beta_2] = 0_A$, then there exists an abelian atom $\gamma \in \operatorname{Con} \mathbf{A}$ such that $[\alpha_1,\beta_1],[\alpha_1,\beta_2] \leq \gamma$.*

Proof. We essentially follow the proof of [19, Lemma 4.20]. As $\mathcal{V}$ satisfies C1, the claim is true in $\mathcal{V}$ by Proposition 3.3. Thus it will suffice to show that the claim can be formulated as a first-order sentence. The claim is equivalent to the following statement:

> For all $a_0,b_0,a_1,b_1,c_1,d_1,c_2,d_2$, letting $\alpha_0 = \operatorname{Cg}^{\mathbf{A}}(a_0,b_0)$, $\overline{\alpha}_0 = \operatorname{Cg}^{\mathbf{A}}(a_1,b_1)$, $\alpha_1 = \alpha_0 \vee \overline{\alpha}_0$, $\beta_1 = \operatorname{Cg}^{\mathbf{A}}(c_1,d_1)$, and $\beta_2 = \operatorname{Cg}^{\mathbf{A}}(c_2,d_2)$, if:
> (1) α_0 is an abelian atom;
> (2) α_1 is abelian;
> (3) α_1/α_0 is an abelian atom in $\operatorname{Con}(\mathbf{A}/\alpha_0)$;
> (4) $[\alpha_0,\beta_1] = [\alpha_0,\beta_2] = 0_A$;
> then there exist e,f such that, setting $\gamma = \operatorname{Cg}^{\mathbf{A}}(e,f)$,
> (5) γ is an abelian atom;
> (6) $[\alpha_1,\beta_1] \leq \gamma$ and $[\alpha_1,\beta_2] \leq \gamma$.

(1) and (5) are first-order by Corollary 7.9(2a), (2) is equivalent to $\mu(a_0,b_0)$ & $\mu(a_1,b_1)$ by Lemma 7.12, and (4) is equivalent to $\Omega(a_0,b_0,c_1,d_1)$ & $\Omega(a_0,b_0,c_2,d_2)$. Since $|A/\operatorname{ann}(a_0,b_0)| \leq r$ by Corollary 7.9(1), Lemma 2.7 implies that α_0 is definable by the formula $\theta(x,y,a_0,b_0)$. (3) can now be stated by asserting $\operatorname{AbAt}(a_1,b_1)$ "mod α_0." By this we mean taking the formula $\operatorname{AbAt}(x,y)$ and replacing every occurrence of an equality $u = v$ with $\theta(u,v,a_0,b_0)$.

It remains to show that (6) can be formulated as a first-order statement. Let $\beta = \operatorname{Cg}^{\mathbf{A}}(c,d)$ be any principal congruence of $\mathbf{A}$ satisfying $[\alpha_0,\beta] = 0_A$.

Claim: $[\alpha_1,\beta] \leq \gamma$ iff

(a) $[\overline{\alpha}_0,\beta] = 0_A$, or
(b) in $\mathbf{A}/\gamma$, $C(a_0/\gamma,b_0/\gamma,c/\gamma,d/\gamma)$ & $C(a_1/\gamma,b_1/\gamma,c/\gamma,d/\gamma)$.

Proof of Claim. ($\Leftarrow$) If (a) holds, then $[\alpha_1,\beta] = [\alpha_0 \vee \bar{\alpha},\beta] = 0_A$ by semi-distributivity. If (b) holds, then $C(\alpha_0 \vee \gamma,\beta \vee \gamma;\gamma)$ and $C(\overline{\alpha}_0 \vee \gamma,\beta \vee \gamma;\gamma)$ hold (this is equivalent to (b)), so $C(\alpha_0 \vee \overline{\alpha}_0,\beta;\gamma)$, so $[\alpha_1,\beta] \leq \gamma$.

($\Rightarrow$) Assume $[\alpha_1, \beta] \leq \gamma$. Then either $[\alpha_1, \beta] = 0_A$ or $[\alpha_1, \beta] = \gamma$. If $[\alpha_1, \beta] = 0_A$ then (a) holds. Assume $[\alpha_1, \beta] = \gamma$. Then $\gamma \leq \beta$ and $C(\alpha_1 \vee \gamma, \beta; \gamma)$. These facts imply $C(\alpha_0 \vee \gamma, \beta \vee \gamma; \gamma)$ and $C(\overline{\alpha}_0 \vee \gamma, \beta \vee \gamma; \gamma)$, so (b) holds, proving the Claim. $\square$

Returning to the proof of (6), observe that γ (like α_0 considered above) is definable by the formula $\theta(x, y, e, f)$. It follows from the Claim that we can express $[\alpha_1, \beta_i] \leq \gamma$ by asserting

$$\Omega(a_1, b_1, c_i, d_i) \text{ or } [\ \Omega(a_0, b_0, c_i, d_i) \text{ “mod } \gamma\text{” and } \Omega(a_1, b_1, c_i, d_i) \text{ “mod } \gamma\text{”}]$$

where by $\Omega(x, y, z, w)$ “mod γ” we mean the formula obtained from $\Omega(x, y, z, w)$ by replacing each occurrence of an equality $u = v$ with $\theta(u, v, e, f)$. This shows that (6) is expressible as a first-order statement, and completes the proof of the Lemma. $\square$

The remainder of the proof departs from McKenzie's proof for the congruence modular case.

Lemma 7.19. *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$, α is an abelian atom in* $\operatorname{Con} \mathbf{A}$*, β is a principal congruence, and $[\alpha, \beta] = 0_A$, then:*

(1) *If $\lambda \in \operatorname{Con} \mathbf{A}$ satisfies $[\alpha, \lambda] = 0_A$ and $C(\lambda, \alpha \vee \beta; \alpha)$, then $[\lambda, \beta] = 0_A$.*
(2) *$|A/(0_A : \beta)| \leq r^2$ if $\alpha \prec \alpha \vee \beta$.*

Proof. (1) It suffices to prove the claim under the assumption that λ is principal. Let $\alpha = \mathrm{Cg}^{\mathbf{A}}(a, b)$, $\beta = \mathrm{Cg}^{\mathbf{A}}(c, d)$, and $\lambda = \mathrm{Cg}^{\mathbf{A}}(u, v)$. The claim is then equivalent to the following:

If $\mathrm{AbAt}(a, b)$, $\Omega(a, b, c, d)$, $\Omega(a, b, u, v)$ and “$\Omega(u, v, c, d)$ mod α,” then $\Omega(u, v, c, d)$.

All but the last of the hypotheses is clearly first-order, and the last (“$\Omega(u, v, c, d)$ mod α”) can also be expressed by a first-order formula since α is a definable congruence. Hence it suffices to prove the claim in $\mathcal{V}$. Assume $\mathbf{A} \in \mathcal{V}$. The hypotheses imply $[\alpha, \beta \vee \lambda] = 0_A$ and $[\lambda, \beta] \leq \alpha$. Thus if $[\lambda, \beta] \neq 0_A$ then $[\lambda, \beta] = \alpha$, so $\alpha \leq [\beta \vee \lambda, \beta \vee \lambda]$, so $\alpha = [\alpha, \beta \vee \lambda]$ by C1, contradiction.

(2) By Corollary 7.9(1), $(0_A : \alpha)$ and $(\alpha : \alpha \vee \beta)$ both have index at most r. Thus it will suffice to $(0_A : \alpha) \cap (\alpha : \alpha \vee \beta) \subseteq (0_A : \beta)$. Let λ be a principal congruence contained in $(0_A : \alpha) \cap (\alpha : \alpha \vee \beta)$; it suffices to prove $\lambda \leq (0_A : \beta)$. We did this in (1). $\square$

Recall [7, Definition 7.1] that a 2-*snag* in an algebra $\mathbf{A}$ is a pair $(c, d) \in A^2$ with $c \neq d$ for which there exists $f \in \mathrm{Pol}_2(\mathbf{A})$ satisfying $f(c, d) = f(d, c) = f(c, c) = c$ and $f(d, d) = d$. Such an f is called a *pseudo-meet operation* for the 2-snag. Note that if (c, d) is a 2-snag and $\beta = \mathrm{Cg}^{\mathbf{A}}(c, d)$, then $[\beta, \beta] = \beta$.

Lemma 7.20. *For all sufficiently large j, suppose $\mathbf{A} \in \mathcal{V}^{(j)}$ is finite, $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ with $\alpha \prec \beta$, and β/α is non-abelian. Then β contains a 2-snag having a pseudo-meet operation in* $\mathrm{Pol}_2^{(r)}(\mathbf{A})$.

Proof. Let $\gamma = (\alpha : \beta)$. Then $\gamma/\alpha = (0_{A/\alpha} : \beta/\alpha)$, so γ has index at most r by Lemma 7.9(1). Moreover, γ is the largest congruence of $\mathbf{A}$ containing α but not β (because β/α is nonabelian). Let θ be its unique upper cover. Then (γ, θ) is perspective to (α, β), so θ/γ is non-abelian. Pick $e_1, \ldots, e_r \in A$ so that $\{e_1, \ldots, e_r\}$ contains a transversal for γ. Let $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(a, b, e_1, \ldots, e_r)$. The key observation is that the map $\mathbf{A}/\gamma \to \mathbf{B}/\gamma{\restriction}_B$ given by $a/\gamma \mapsto a/\gamma \cap B$ is an isomorphism. Hence $\gamma{\restriction}_B \prec \theta{\restriction}_B$ in $\mathrm{Con}\,\mathbf{B}$ and $\theta{\restriction}_B$ is nonabelian over $\gamma{\restriction}_B$.

Observe that we still have $(a, b) \notin \gamma{\restriction}_B$ but $(a, b) \in \beta' := \mathrm{Cg}^{\mathbf{B}}(a, b) \leq \beta{\restriction}_B \leq \theta{\restriction}_B$. Let $\psi \in \mathrm{Con}\,\mathbf{B}$ be an upper cover of $\alpha' := \alpha{\restriction}_B \cap \beta'$ below β'. Then (α', ψ) and $(\gamma{\restriction}_B, \theta{\restriction}_B)$ are perspective, so ψ is non-abelian over α'. Let (c, d) be a 2-snag of $\mathbf{B}$ in $\psi \setminus \alpha'$ (this exists by tame congruence theory; see [7, Exercise 5.11(1)]); then (c, d) satisfies the claim. $\square$

Definition 7.21. Given a variety $\mathcal{V}$ with a difference term p, an algebra $\mathbf{A} \in \mathcal{V}$, and $a, b \in A$, we call (a, b) a *Maltsev pair* (for p) if $p(a, b, b) = a$. Given an algebra $\mathbf{A}$, $a, b, c, d \in A$, and $f \in \mathrm{Pol}_1(\mathbf{A})$, we write $(c, d) \xrightarrow{f} (a, b)$ to mean $f(c) = a$ and $f(d) = b$.

Lemma 7.22. *Assume that $\mathcal{V}$ is a variety with difference term p, $\mathbf{A} \in \mathcal{V}$, $a, b, c, d \in A$, (a, b) is a Maltsev pair, and* $\mathrm{ann}(c, d)$ *has finite index k. If $(c, d) \xrightarrow{f} (a, b)$ for some polynomial f, then $(c, d) \xrightarrow{g} (a, b)$ for some polynomial $g \in \mathrm{Pol}_1^{(k+3)}(\mathbf{A})$.*

Proof. Choose a term $t(x, y_1, \ldots, y_m)$ and parameters $\mathbf{u} \in A^m$ so that $f(x) = t^{\mathbf{A}}(x, \mathbf{u})$. Let T be a transversal for $\mathrm{ann}(c, d)$. For each u_i let e_i be the unique member of T which is $\mathrm{ann}(c, d)$-related to u_i and let $f'(x) = t^{\mathbf{A}}(x, \mathbf{e})$. Then $f' \in \mathrm{Pol}_1^{(k)}(\mathbf{A})$.

We have

$$p(f(c), f(c), f(d)) = f(d) = p(f'(c), f'(c), f(d)),$$

so

$$p(f(c), f(d), f(d)) = p(f'(c), f'(d), f(d)),$$

i.e.,

$$a = p(a, b, b) = p(f'(c), f'(d), f(d)).$$

It follows that

$$g(x) := p(a, p(f'(c), f'(x), f(d)), b) = p(a, p(f'(c), f'(x), b), b)$$

witnesses that $(c, d) \xrightarrow{g} (a, b)$. Since the polynomial g involves only the k parameters of f' along with the three parameters $a, b, f'(c)$ we get that $g \in \mathrm{Pol}_1^{(k+3)}(\mathbf{A})$. $\square$

Lemma 7.23. *For all sufficiently large j, if $\mathbf{A} \in \mathcal{V}^{(j)}$ is finite and $0_A \prec \alpha \prec \beta$ in* $\mathrm{Con}\,\mathbf{A}$ *with α abelian and $[\beta, \beta] = \beta$, then $[\alpha, \beta] = \alpha$.*

Proof. Assume instead that $[\alpha, \beta] = 0_A$. If there exists $\gamma \prec \beta$ with $\gamma \neq \alpha$, then β/γ would be perspective with $\alpha/0_A$, so would be abelian, implying $[\beta, \beta] \leq \gamma$ which is false. Thus α is the unique lower cover of β.

As β/α is non-abelian, there exists a 2-snag $(c, d) \in \beta$ having a pseudo-meet operation $h \in \mathrm{Pol}_2^{(r)}(\mathbf{A})$, by Lemma 7.20. Clearly $(c, d) \notin \alpha$. Thus $\mathrm{Cg}^{\mathbf{A}}(c, d) = \beta$.

Let U be a $(0_A, \alpha)$-minimal set, let $e(x) \in \mathrm{Pol}_1(\mathbf{A})$ satisfy $e^2(x) = e(x)$ and $e(A) = U$, let V be a trace in U, and choose $(\overline{a}, \overline{b}) \in V^2 \setminus 0_V$. Because U has empty tail, $\mathbf{A}|_U$ is solvable by [7, Theorems 4.31 and 4.32].

$(\overline{a}, \overline{b}) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$, so there exists a chain $\overline{a} = a_0, a_1, \ldots, a_n = \overline{b}$ of elements of U and polynomials $f_1, \ldots, f_n \in \mathrm{Pol}_1(\mathbf{A})$ such that $\{a_{i-1}, a_i\} = \{ef_i(c), ef_i(d)\}$ for $1 \leq i \leq n$. If for some i we have $(a_{i-1}, a_i) \notin \alpha$, then $ef_i(\beta|_N) \not\subseteq \alpha$, so $ef_i(N)$ is itself an (α, β)-trace, so contains a 2-snag. But $ef_i(N) \subseteq U$ which is solvable, contradiction. Thus $a_0, \ldots, a_n \in V$. This proves that $(c, d) \xrightarrow{f} (a, b)$ for some $(a, b) \in V^2 \setminus 0_A$ and some $f \in \mathrm{Pol}_1(\mathbf{A})$. As $\mathrm{ann}(c, d)$ has index at most r^2 by Lemma 7.19, we have $(c, d) \xrightarrow{g} (a, b)$ for some $g \in \mathrm{Pol}_1^{(r^2+3)}(\mathbf{A})$, by Lemma 7.22.

In summary, we have elements a, b, c, d in an algebra $\mathbf{A} \in \mathcal{V}^{(j)}$ satisfying:

- $a \neq b$;
- (c, d) is a 2-snag having a pseudo-meet operation in $\mathrm{Pol}_1^{(r)}(\mathbf{A})$;
- $(c, d) \xrightarrow{g} (a, b)$ for some $g \in \mathrm{Pol}_1^{(r^2+3)}(\mathbf{A})$;
- $\Omega(a, b, c, d)$.

Modulo σ_{r+1} & σ_{r^2+4}, this configuration is first-order definable. It cannot exist in $\mathcal{V}$ (for if $\alpha := \mathrm{Cg}^{\mathbf{A}}(a, b)$ and $\beta := \mathrm{Cg}^{\mathbf{A}}(c, d)$, then the configuration implies $0_A < \alpha \leq \beta = [\beta, \beta]$ and $[\alpha, \beta] = 0_A$, which violates C1). Hence it cannot exist in $\mathcal{V}^{(j)}$ for sufficiently large j. □

Corollary 7.24. *$\mathcal{V}^{(j)}$ satisfies* C1 *for all sufficiently large j.*

Proof. Proposition 3.3 with Lemmas 7.10, 7.12, 7.17, 7.18, and 7.23. □

We can now prove that $\mathcal{V}$ is finitely based. Choose j large enough so that all of the foregoing claims about $\mathcal{V}^{(j)}$ are satisfied. Then $\mathcal{V}^{(j)}$ is locally finite, satisfies C1, and is such that for every $\mathbf{A} \in \mathcal{V}^{(j)}$ and atom $0_A \prec \alpha$, the index of $(0_A : \alpha)$ is at most r. It follows by Proposition 3.2 that $\mathcal{V}^{(j)}$ has a finite residual bound. Then in the usual way we can argue that $\mathcal{V}$ is finitely axiomatizable relative to $\mathcal{V}^{(j)}$. Since $\mathcal{V}^{(j)}$ is finitely based, so is $\mathcal{V}$.

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