

Essential arities of term operations in finite algebras

Ross Willard¹

Department of Pure Mathematics, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 29 June 1993; revised 30 June 1994

Abstract

Given an algebra A , $p_n(A)$ denotes the number of distinct n -ary term operations $t: A^n \rightarrow A$ of A which depend on all n variables. We solve some problems of Berman, Grätzer and Kisielewicz concerning the sequence $\langle p_0(A), p_1(A), \dots, p_n(A), \dots \rangle$ in case $|A|$ is finite. Our methods yield new results about totally symmetric functions on a finite set.

0. Introduction

A clone on a nonempty set A is a set of finitary operations $f: A^n \rightarrow A$ ($n < \omega$) which contains all of the projections and is closed under composition. A natural example is the clone $\text{Clo } A$ of all term operations of an algebra A , and in fact every clone is realized in this way.

If C is a clone on A and $n \geq 1$, then $E_n(C)$ denotes the subset of C consisting of those n -ary operations which depend on all of their variables. $E_0(C)$ is defined to be the set of all unary operations in C which are constant-valued, and (following [5,2]) $p_n(C)$ denotes $|E_n(C)|$ for $n \geq 0$. The p_n -sequence of C is $\langle p_n(C) \rangle_{n < \omega}$. If A is an algebra then we extend the above notation by letting $E_n(A) = E_n(\text{Clo } A)$ and $p_n(A) = p_n(\text{Clo } A)$.

A natural question to ask is which sequences $\bar{p} = \langle p_n \rangle_{n < \omega}$ of nonnegative integers are representable as the p_n -sequence of some clone C . Some deep results characterizing such \bar{p} in case $p_0 = 0$ and $p_1 > 1$ [7], and characterizing the zeros in \bar{p} in case $p_0 = 0$ and $p_1 = 1$ [10], are known. On the other hand, it is not too hard to show that any sequence \bar{p} satisfying $p_0 > 0$ and $p_1 > 0$ is representable [6].

The proofs of these results typically require infinite algebras whose fundamental operations have arbitrarily large arities. Therefore several authors have posed the

¹ This research was supported by the NSERC of Canada.

corresponding questions for the p_n -sequences of algebras which are finite, are of finite type, or whose fundamental operations have bounded arity (see e.g. [3, 8, 2, 5]). In this paper we prove a few results concerning the p_n -sequences of finite algebras. To be precise, let \bar{p} be the p_n -sequence of some finite algebra and let $S = \{n \in \mathbb{N} : p_n^* > 0\}$, where $p_n^* = p_n$ if $n \neq 1$, $p_1^* = p_1 - 1$. We show that: (1) either \bar{p} is bounded or $p_n \geq n$ for sufficiently large n ; (2) S is equal modulo a finite set to \emptyset , \mathbb{N} , or the set of odd positive integers; (3) in the last case, $\bar{p} = \langle \dots, 0, N, 0, N, 0, N, \dots \rangle$ eventually. We also characterize those subsets $S \subseteq \mathbb{N}$ which arise in this way, solving a problem of Grätzer and Kisielewicz. We refute a conjecture of Berman concerning the eventual behavior of \bar{p} , and prove several results concerning totally symmetric operations on a finite set.

1. Preliminaries

We shall be concerned with functions $f: A^V \rightarrow B$ where A, B, V are finite nonempty sets. We refer to elements of V as the *variables* of f ; elements of A^V are *assignments* to the variables, and are written as vectors $\bar{a} = (a_x)_{x \in V}$. If a linear ordering of V is specified, say $V = \{x_1, \dots, x_n\}$, then we use the usual notation $f(a_1, \dots, a_n)$ and $f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ to indicate values of f and functions derived from f , respectively.

$x \in V$ is an *essential* variable of f , and f *depends on* x , if there exist $\bar{a}, \bar{b} \in A^V$ such that $a_y = b_y$ for all $y \in V \setminus \{x\}$ but $f(\bar{a}) \neq f(\bar{b})$; otherwise x is *inessential*. $|V|$ is the *arity* of f ; the *essential arity* of f is the number of its essential variables.

Suppose θ is an equivalence relation on V , with $v: V \rightarrow V/\theta$ being the natural map. We let f_θ denote the function $A^{(V/\theta)} \rightarrow B$ given by $f_\theta(\bar{a}) = f(\bar{a} \circ v)$, and say that f_θ is obtained from f by *collapsing the variables* through θ . We informally identify V/θ with any one of its choice sets, so that the variables of f_θ are among the variables of f . If x, y, z, w are distinct elements of V then f_{xy} , f_{xyz} , and $f_{xy,zw}$ denote the functions f_θ where $\theta = 0_V \cup \{x, y\}^2$, $0_V \cup \{x, y, z\}^2$, and $0_V \cup \{x, y\}^2 \cup \{z, w\}^2$, respectively. A fact we shall use repeatedly is this: if $\theta \subseteq \phi$ and f_ϕ depends on x , then f_θ depends on at least one variable in the ϕ -class of x .

To each function $f: A^V \rightarrow B$ which depends on all its variables, we build an edge-colored graph (V, E, χ) on the variable set as follows. If $x, y \in V$ are distinct, then $\{x, y\} \in E$ iff f_{xy} depends on all of its variables except possibly $x (= y)$ and at most one other variable. If $e = \{x, y\} \in E$, then $\chi(e)$ is defined by

$$\chi(e) = \begin{cases} \square & \text{if } f_{xy} \text{ depends on all of its variables,} \\ \circ & \text{if } f_{xy} \text{ depends on all of its variables except } x (= y), \\ \boxplus & \text{if } f_{xy} \text{ depends on all of its variables except } z \ (z \notin \{x, y\}), \\ \odot & \text{if } f_{xy} \text{ depends on all of its variables except } x (= y) \text{ and } z. \end{cases}$$

We shall write $x \square y$ (or $x \odot y$, etc.) to mean that $e = \{x, y\} \in E$ and $\chi(e)$ has the indicated value. We also use \odot to denote \circ or \odot , $z \in V$.

Lemma 1.1. Suppose $f: A^V \rightarrow B$ depends on all of its variables.

1. If $x \circ y$ and $z \notin \{x, y\}$, then $\{x, z\} \in E$. Moreover, either $x \sqsubseteq z$ or $x \circ z$ or $x \sqsupseteq z$.
2. If $x \sqsupseteq y$ then $\{x, z\} \in E$. Moreover, either $x \sqsubseteq z$ or $x \circ z$ or $x \sqsupseteq z$.
3. If $x \sqsupseteq y$ and $w \notin \{x, y, z\}$, then $\{z, w\} \in E$. Moreover, either $z \sqsubseteq w$ or $z \sqsupseteq x$ or $z \sqsupseteq y$ or $z \sqsupseteq w$.
4. There do not exist distinct x, y, z, w with $x \oplus y$ and $z \oplus w$.
5. Suppose $x \sqsupseteq y \sqsupseteq x \sqsupseteq z \sqsupseteq y \sqsupseteq x$ and $w \notin \{x, y, z\}$. Then $x \sqsubseteq w$ and $y \sqsubseteq w$ and $z \sqsubseteq w$.

Proof. Write $V = \{x, y, z, u_1, \dots, u_m\}$.

(1) By hypothesis, $f(x, x, z, \bar{u}) = f_{xy}(x, z, \bar{u}) = h(z, \bar{u})$, where h depends on all of its variables. Note that $f_{xyz}(x, \bar{u}) = f(x, x, x, \bar{u}) = h(x, \bar{u})$, so f_{xyz} depends on all of its variables. Hence $f_{xz}(x, y, \bar{u})$ must depend on u_1, \dots, u_m and either $x (= z)$ or y (or both). So $\{x, z\} \in E$ and either $x \sqsubseteq z$ or $x \circ z$ or $x \sqsupseteq z$.

(2) By hypothesis, $f_{xy}(x, z, \bar{u}) = h(x, \bar{u})$, where h depends on all of its variables, so again f_{xyz} depends on all of its variables. The rest of the argument is the same as in (1).

(3) Rewrite $V = \{x, y, z, w, \bar{u}\}$. By hypothesis, $f(x, x, z, w, \bar{u}) = h(x, w, \bar{u})$, where h depends on all of its variables. Then $f(x, x, z, z, \bar{u}) = h(x, z, \bar{u})$, so f_{xyzw} depends on all of its variables. Hence f_{zw} depends on $z (= w)$, \bar{u} , and either x or y (or both).

(4) Assuming $x \oplus y$, we have $f(x, x, z, w, \bar{u}) = h(w, \bar{u})$, where h depends on all of its variables. This time $f_{xyzw}(x, z, \bar{u}) = h(z, \bar{u})$, so f_{xyzw} depends on z . Hence f_{zw} must also depend on z , and thus $\{z, w\}$ cannot be labelled by \oplus .

(5) Suppose for example we do not have $z \sqsubseteq w$. Then since $x \sqsupseteq y$, the previous item gives either $z \sqsupseteq x$ or $z \sqsupseteq y$ or $z \sqsupseteq w$. If the former holds, then item (2) implies that $z \sqsubseteq x$ or $z \circ x$ or $z \sqsupseteq x$, all of which are false. A similar argument works in the other case. \square

We end this section by proving the key result on which this paper rests.

Lemma 1.2. Suppose $f: A^V \rightarrow B$ where A, B, V are finite nonempty sets and f depends on all of its variables. Let $k = |A|$ and $n = |V|$. If $n > k$ then there exist distinct $x, y \in V$ such that f_{xy} has at most one inessential variable.

Proof. We may assume $k \geq 2$. Let r be the maximum of the essential arities of all the f_{xy} , $x, y \in V$. Assume $r < n - 2$. We shall find a contradiction.

Claim 1. There exist $u, v \in V$ such that f_{uv} is essentially r -ary and does not depend on $u (= v)$.

Indeed, suppose this were not the case. Then $n \geq 4$. Choose y, z such that f_{yz} is essentially r -ary and depends on $y (= z)$. Let $s = n - r \geq 3$, and write $V = \{x_1, \dots, x_n\}$

where $y = x_1$ and $z = x_n$ and

$$f(y, x_2, x_3, \dots, x_{s+1}, \dots, x_{n-1}, y) = h(x_{s+1}, \dots, x_{n-1}, y),$$

h depending on all of its variables. Let $u = x_2$ and $v = x_3$. Then

$$f(y, u, u, \dots, x_{s+1}, \dots, x_{n-1}, y) = h(x_{s+1}, \dots, x_{n-1}, y)$$

so $f_{yz, uv}$ depends on all of its variables, hence f_{uv} depends on x_{s+1}, \dots, x_{n-1} , and either y or z (at least). By the maximality of r , f_{uv} cannot depend on any variables other than these, so in particular does not depend on u , which proves Claim 1.

Now fix u, v as in Claim 1, and rewrite $V = \{x_1, \dots, x_n\}$ so that $u = x_{n-1}$, $v = x_n$, and

$$f(x_1, \dots, x_{n-2}, u, u) = h(x_1, \dots, x_r), \quad (1)$$

h depending on all of its variables.

Claim 2. For all i, j with $1 \leq i < j \leq n$, if $a_1, \dots, a_n \in A$ satisfy $a_i = a_j$ then

$$f(a_1, \dots, a_n) = h(a_1, \dots, a_r). \quad (2)$$

The claim will be proved by considering cases.

Case 1: $r < i, j$. The claim is given to be true if $(i, j) = (n-1, n)$. If $(i, j) \neq (n-1, n)$, then note that $a_i = a_j = a_{n-1} = a_n$ implies (2), so $f_{x_i x_j}$ depends on x_1, \dots, x_r at least, hence on no other variables (by the maximality of r). Thus $a_i = a_j$ implies

$$\begin{aligned} f(a_1, \dots, a_r, a_{r+1}, \dots, a_{n-1}, a_n) &= f(a_1, \dots, a_r, u, \dots, u, u) \\ &= h(a_1, \dots, a_r) \quad \text{by (1).} \end{aligned}$$

Case 2: $i \leq r < j$. By symmetry we may assume that $i = 1$ and (by case 1) $j = n$. Clearly $a_1 = a_{n-1} = a_n$ implies (2) by case 1, so $f_{x_1 x_n}$ depends on x_2, \dots, x_r at least, and hence cannot depend on both of x_{n-2} and x_{n-1} . Suppose with no loss of generality that $f_{x_1 x_n}$ does not depend on x_{n-1} . Then

$$\begin{aligned} f(a_1, \dots, a_r, \dots, a_{n-1}, a_1) &= f(a_1, \dots, a_r, \dots, a_1, a_1) \\ &= h(a_1, \dots, a_r) \quad \text{by case 1.} \end{aligned}$$

Case 3: $i, j \leq r$. By the symmetry due to cases 1 and 2 we may assume that $(i, j) = (1, 2)$. Assume first that, for some $t > r$, $f_{x_1 x_2}$ does not depend on x_t . By symmetry, assume that $t = n$. Then

$$\begin{aligned} f(a_1, a_1, a_3, \dots, a_r, \dots, a_{n-1}, a_n) &= f(a_1, a_1, a_3, \dots, a_r, \dots, a_{n-1}, a_1) \\ &= h(a_1, a_1, a_3, \dots, a_r) \quad \text{by case 2.} \end{aligned}$$

Next assume that $f_{x_1 x_2}$ depends on all of x_{r+1}, \dots, x_n . Since $f_{x_1 x_2}$ can depend on at most $n-3$ variables we have $f_{x_1 x_2}$ does not depend on x_t for some t , $3 \leq t \leq r$. Say

$t = 3$ for simplicity. Then $h_{x_1x_2}$ also does not depend on x_3 , so

$$\begin{aligned} f(a_1, a_1, a_3, a_4, \dots, a_r, \dots, a_n) &= f(a_1, a_1, a_n, a_4, \dots, a_r, \dots, a_n) \\ &= h(a_1, a_1, a_n, a_4, \dots, a_r) \quad (\text{case 2}) \\ &= h(a_1, a_1, a_3, a_4, \dots, a_r). \end{aligned}$$

Thus $a_1 = a_2$ implies (2) under either the first or the second assumption. This completes the proof of case 3, and of the claim.

To finish the proof of the lemma, note that $n > |A|$ so for every $\bar{a} \in A^n$ there exists $i < j$ such that $a_i = a_j$. Hence Claim 2 implies that f is essentially r -ary, which is a contradiction. \square

2. Totally symmetric operations

Throughout this section, A, B, V are finite nonempty sets, $f: A^V \rightarrow B$ depends on all of its variables, and $n = |V|$, $k = |A|$. f is said to be *totally symmetric* if for all $\bar{a} \in A^V$ and all permutations σ of V , $f(\bar{a} \circ \sigma) = f(\bar{a})$.

Theorem 2.1. *Suppose $n > \max(k, 3)$ and there do not exist distinct $x, y \in V$ such that f_{xy} depends on all of its variables. Then f is totally symmetric, and for all distinct $x, y \in V$, f_{xy} depends on all of its variables except $x (= y)$.*

Proof. We shall use the graph (V, E, χ) defined in the previous section. The hypothesis asserts that there are no edges labelled by \square .

Claim 1. *There is an edge labelled by \bigcirc .*

For suppose this were not true. By Lemma 1.2, there exist x, y such that f_{xy} has at most one inessential variable. Then it must be labelled by \boxed{z} for some $z \notin \{x, y\}$. Using Lemma 1.1(2) we see that $x \boxed{y} z$ and $y \boxed{x} z$ as well. Pick $w \notin \{x, y, z\}$ (which can be done as $n > 3$). Then Lemma 1.1(5) implies $x \square w$, a contradiction.

Claim 2. *Suppose $x \bigcirc y$. Then there exists $z \notin \{x, y\}$ such that $x \bigcirc z$ or $y \bigcirc z$.*

Suppose this were false. Then for each $z \notin \{x, y\}$ Lemma 1.1(1) implies $x \boxed{y} z$ and $y \boxed{x} z$. Write $V = \{x_1, \dots, x_n\}$ with $x = x_1$ and $y = x_2$. Pick $h: A^{n-2} \rightarrow B$ so that $f(x, x, x_3, \dots, x_n) = h(x_3, \dots, x_n)$.

As in the proof of Lemma 1.2 we shall obtain a contradiction by showing that for all i, j with $1 \leq i < j \leq n$, if $\bar{a} \in A^n$ and $a_i = a_j$ then

$$f(\bar{a}) = h(a_3, \dots, a_n). \quad (3)$$

By assumption, the assertion is true when $(i, j) = (1, 2)$. Suppose $i = 1$ or 2 but $(i, j) \neq (1, 2)$. By symmetry we may assume $(i, j) = (1, 3)$. Let $z = x_3$. Then

$$\begin{aligned} f(x, y, x, \bar{w}) &= f(x, x, x, \bar{w}) \quad \text{since } x \boxed{y} z \\ &= h(x, \bar{w}). \end{aligned}$$

Finally, assume that $i \geq 3$. By symmetry we can assume that $(i, j) = (3, 4)$. Put $z = x_3$ and $w = x_4$.

Case 1: f_{zw} does not depend on x . Then

$$\begin{aligned} f(x, y, z, z, \bar{u}) &= f(z, y, z, z, \bar{u}) \\ &= f(z, z, z, z, \bar{u}) \quad (\text{since } x \boxed{y} z) \\ &= h(z, z, \bar{u}). \end{aligned}$$

One can argue similarly if f_{zw} does not depend on y .

Case 2: f_{zw} depends on both x and y . Note that if $z \circ w$ then Lemma 1.1(1) would contradict $x \boxed{y} z$. Since f_{zw} cannot depend on all of its variables (by hypothesis), there must exist t such that $5 \leq t \leq n$ and f_{zw} does not depend on x_t . Suppose with no loss of generality that $t = 5$. Let $u = x_5$. Then $f(x, y, z, z, u, \bar{v}) = f(x, y, z, z, x, \bar{v}) = h(z, z, x, \bar{v})$, where the last equation follows from the truth of (3) when $(i, j) = (1, 5)$. A similar argument shows that $f(x, y, z, z, u, \bar{v}) = h(z, z, y, \bar{v})$. This shows that $h(z, z, u, \bar{v})$ does not depend on u , and hence

$$\begin{aligned} f(x, y, z, z, u, \bar{v}) &= h(z, z, x, \bar{v}) \\ &= h(z, z, u, \bar{v}) \end{aligned}$$

which establishes (3) in this last case.

Claim 3. $u \circ v$ for all distinct $u, v \in V$.

By the previous two claims we may begin with distinct x, y, z satisfying $x \circ y$ and $x \circ z$. For any $w \notin \{x, y, z\}$ we can use Lemma 1.1(1) to obtain $x \circ w$. Now suppose u, v are distinct and both different from x . Pick $w \notin \{x, u, v\}$ (using $n \geq 4$). On the one hand, we know that $x \circ u$, so Lemma 1.1(1) implies that either $u \circ v$ or $u \boxed{x} v$. On the other hand, we know that $x \circ w$, so Lemma 1.1(3) rules out the possibility that $u \boxed{x} v$.

Claim 4. f is totally symmetric.

To see this, fix distinct $x, y \in V$, write $V = \{x_1, \dots, x_n\}$ with $x = x_1$ and $y = x_2$, and choose $h: A^{n-2} \rightarrow B$ so that $f(x, x, \bar{u}) = h(\bar{u})$. First we show that h is totally symmetric. By symmetry of the assumptions, it suffices to show that $h(u, v, \bar{w}) = h(v, u, \bar{w})$.

In fact,

$$\begin{aligned}
 h(u, v, \bar{w}) &= f(u, u, u, v, \bar{w}) \\
 &= f(v, u, v, v, \bar{w}) \quad \text{since } x_1 \circ x_3 \\
 &= f(v, u, u, u, \bar{w}) \quad \text{since } x_3 \circ x_4 \\
 &= f(v, v, v, u, \bar{w}) \quad \text{since } x_2 \circ x_3 \\
 &= h(v, u, \bar{w}).
 \end{aligned}$$

Next we shall show that for all i, j with $1 \leq i < j \leq n$, if $\bar{a} \in A^n$ satisfies $a_i = a_j$, then

$$f(\bar{a}) = h(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n). \quad (4)$$

Since (4) is already known to be true for $(i, j) = (1, 2)$, it will be enough to prove that if $i < j$ and $i' < j'$ and $\{i, j\} \cap \{i', j'\} \neq \emptyset$, then the truth of (4) for (i, j) implies the truth of (4) for (i', j') . By the total symmetry of h we may reorder V so that $i = i' = 1$, $j = 2$, and $j' = 3$. Let $u = x_1$, $v = x_2$, and $w = x_3$. By assumption, $f(u, u, w, \bar{z}) = h(w, \bar{z})$. Hence $f(u, v, u, \bar{z}) = f(v, v, v, \bar{z}) = h(v, \bar{z})$ as desired.

Finally, suppose $\bar{a} \in A^n$ and $\sigma \in S_n$. As $n > k$ there exists $i < j$ such that $a_i = a_j$. By the previous discussion, both $f(a_1, \dots, a_n)$ and $f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ are equal to h evaluated at $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ in any order. Hence they are equal to each other. This completes the proof of Claim 4. Claims 3 and 4 yield the theorem. \square

We can sharpen this last theorem by introducing some terminology and results from [2]. Let $\text{Sub}(A)$ denote the collection of all subsets of A . Define $\text{supp} : A^V \rightarrow \text{Sub}(A)$ and $\text{oddsupp} : A^V \rightarrow \text{Sub}(A)$ as follows:

$$\begin{aligned}
 \text{supp}(\bar{a}) &= \{a_v : v \in V\}, \\
 \text{oddsupp}(\bar{a}) &= \{a_v : |\{w \in V : a_w = a_v\}| \text{ is odd}\}.
 \end{aligned}$$

We say that $g : A^V \rightarrow B$ is *determined by supp* (respectively, *by oddsupp*) if there exists a function $g^* : \text{Sub}(A) \rightarrow B$ such that g is equal to $g^* \circ \text{supp}$ (resp. $g^* \circ \text{oddsupp}$). Note that if g is determined by either supp or oddsupp , then g is totally symmetric and hence either is constant or depends on all of its variables. The next lemma is implicit in [2] (in the proofs of Lemmas 2.5 and 2.7).

Lemma 2.2. *Suppose $g : A^V \rightarrow B$ is essentially n -ary and totally symmetric, where $n = |V|$ and $k = |A|$.*

1. *If $n > 2$ and for some (hence any) distinct $x, y \in V$, g_{xy} is essentially $(n-1)$ -ary and totally symmetric, then g is determined by supp .*
2. *Conversely, suppose g is determined by supp . Then g_{xy} is likewise determined by supp for any distinct $x, y \in V$. If $n > k$, then g_{xy} is nonconstant.*

3. If $n > 3$ and for some (hence any) distinct $x, y \in V$, g_{xy} has an inessential variable, then g is determined by *oddsupp*.

4. Conversely, suppose g is determined by *oddsupp*. Then for any distinct $x, y \in V$ the restriction of g_{xy} to $V \setminus \{x, y\}$ is likewise determined by *oddsupp*. If $n > k$, then (this restriction of) g_{xy} is nonconstant.

Thus, recalling the assumptions made at the beginning of this section, we have

Corollary 2.3. Suppose $n > \max(k, 3)$. If no collapse of f is essentially $(n-1)$ -ary, then f is determined by *oddsupp*.

Next we turn to a generalization of Theorem 2.1.

Lemma 2.4. Suppose $n \geq \max(k, 3) + 2$. If f has an essentially $(n-1)$ -ary collapse, then f has an essentially $(n-2)$ -ary collapse.

Proof. Assume instead that f has an essentially $(n-1)$ -ary collapse but no essentially $(n-2)$ -ary collapse. It follows that if f' is an essentially $(n-1)$ -ary collapse of f then no 2-variable collapse of f' depends on all of its variables; hence, f' is determined by *oddsupp* by Corollary 2.3.

Claim 1. If x, y, z, w are distinct and $x \sqsubseteq y$, then (i) either $x \sqsubseteq z$ or $x \odot z$, and (ii) either $z \otimes w$ or $z \odot w$.

To prove this claim, write $V = \{x, y, z, w, \bar{u}\}$ so that $f(x, x, z, w, \bar{u}) = h(x, z, w, \bar{u})$, where h depends on all of its variables, hence is determined by *oddsupp* by hypothesis. Then $f(x, x, x, w, \bar{u}) = h(x, x, w, \bar{u})$, which depends on w and \bar{u} by Lemma 2.2(4). So $f_{xz}(x, y, w, \bar{u})$ depends on w and \bar{u} at least. But f_{xz} cannot depend on exactly $n-2$ variables, so it depends either on both x and y (in which case $x \sqsubseteq z$) or on neither x nor y (in which case $x \odot z$).

Next look at $f(x, x, z, z, \bar{u}) = h(x, z, z, \bar{u})$, which depends on x and \bar{u} . So $f_{zw}(x, y, z, \bar{u})$ depends on \bar{u} and either x or y at least. If f_{zw} depends on x but not y , then since f_{zw} cannot be essentially $(n-2)$ -ary it must not depend on z , and hence $z \odot w$. Similarly, if f_{zw} depends on y but not x then $z \otimes w$. Finally, suppose f_{zw} depends on both x and y . Then it must depend on z as well, and so $f(x, y, z, z, \bar{u}) = h'(x, y, z, \bar{u})$, where h' is determined by *oddsupp* and is essentially $(n-1)$ -ary. Since $n-1 > k$ it follows from Lemma 2.2(4) that h'_{xy} depends on z . But

$$\begin{aligned} h'(x, x, z, \bar{u}) &= f(x, x, z, z, \bar{u}) \\ &= h(x, z, z, \bar{u}) \end{aligned}$$

and $h(x, z, z, \bar{u})$ does not depend on z (since h is determined by *oddsupp*). So this last case is impossible.

Claim 2. *There do not exist distinct x, y, z such that $x \square y$ and $x \square z$.*

Suppose otherwise. For any $w \notin \{x, y, z\}$ we can use the previous claim to obtain $x \square w$. Now suppose u, v are distinct elements of $V \setminus \{x\}$. Since $n \geq 5$ we can choose distinct $u', v' \in V \setminus \{x, u, v\}$. As $x \square u$ it follows from the previous claim that $u \square v$ or $u \otimes v$. Suppose $u \square v$. Again by the previous claim, $x \square u'$ implies $u' \square v'$ or $u' \otimes v'$ on the one hand, while $u \square v$ implies $u' \oplus v'$ or $u' \oplus v'$ on the other. This contradiction proves $u \otimes v$ for all distinct u, v different from x . Since $n - 1 > k$ it follows that f does not depend on x , a contradiction.

Now we complete the proof of the lemma. By hypothesis there exist distinct x, y such that $x \square y$. By Claims 1 and 2 we must have $x \oplus u$ and $y \otimes u$ for all $u \notin \{x, y\}$. Using $n \geq 4$ choose distinct $u, v \in V \setminus \{x, y\}$. Then we have $x \oplus u$ and $y \otimes v$, which contradicts Lemma 1.1(4). \square

Theorem 2.5. *Suppose $n > m \geq \max(k, 3)$ and no collapse of f is essentially m -ary. Then f is determined by oddsupp (and hence n and m have opposite parity).*

Proof. We argue by induction on $n - m$. If $n = m + 1$ then this is just Corollary 2.3. If $n > m + 1$ but f is not determined by oddsupp, then by Corollary 2.3 there must exist distinct x, y such that f_{xy} is essentially $(n - 1)$ -ary. By the previous lemma, f must also have an essentially $(n - 2)$ -ary collapse f' . Thus $n - 2 \neq m$ and so $n - 2 > m$. Therefore the inductive hypothesis may be applied to both f_{xy} and f' , yielding $\text{parity}(n - 1) \neq \text{parity}(m) \neq \text{parity}(n - 2)$, a contradiction. \square

Here is a result which is similar in spirit to Corollary 2.3.

Theorem 2.6. *Suppose $n \geq \max(k, 3) + 2$. If every essentially $(n - 1)$ -ary collapse of f is totally symmetric, then f is determined by either supp or oddsupp.*

Proof. Assume that f is not determined by oddsupp. By Corollary 2.3, some two-variable collapse of f is essentially $(n - 1)$ -ary and thus is totally symmetric. By Lemma 2.2(1) we only need to prove that f is totally symmetric.

For distinct $x, y \in V$ write $x \blacksquare y$ to mean f_{xy} is essentially $(n - 1)$ -ary and is determined by oddsupp.

Claim 1. *If $x \blacksquare y$ and $z \notin \{x, y\}$, then either $x \blacksquare z$ or $x \oplus z$.*

To see this, write $V = \{x, y, z, \bar{u}\}$ and $f(x, x, z, \bar{u}) = h(x, z, \bar{u})$ with h depending on all of its variables and determined by oddsupp. Then $f(x, x, x, \bar{u}) = h(x, x, \bar{u})$, which depends on \bar{u} but not x by Lemma 2.2(4) (as $n - 1 > k$). So $f_{xz}(x, y, \bar{u})$ depends on \bar{u} at least. If y is inessential in f_{xz} then $f(x, y, x, \bar{u}) = f(x, x, x, \bar{u}) = h(x, x, \bar{u})$ and hence $x \oplus z$ as h is determined by oddsupp. A similar argument works if f_{xz} does not

depend on $x (= z)$. Finally, if f_{xz} depends on all of its variables, then $f(x, y, x, \bar{u}) = g(x, y, \bar{u})$, where g is essentially $(n-1)$ -ary and therefore totally symmetric. Furthermore, $g(x, x, \bar{u}) = f(x, x, x, \bar{u}) = h(x, x, \bar{u})$ so g_{xy} does not depend on $x (= y)$. Thus by Lemma 2.2(3), g is determined by oddsupp and so $x \sqsupseteq z$, proving the claim.

Claim 2. *There do not exist x, y such that $x \sqsupseteq y$.*

For suppose $x \sqsupseteq y$. If there does not exist $z \notin \{x, y\}$ such that $x \sqsupseteq z$ or $y \sqsupseteq z$, then by the previous claim, $x \oplus z \oplus y$ for all $z \notin \{x, y\}$. Then choosing distinct $z, w \notin \{x, y\}$ we get $x \oplus z$ and $y \oplus w$, which contradicts Lemma 1.1(4).

So with no loss of generality choose $z \notin \{x, y\}$ such that $x \sqsupseteq z$. The previous claim yields $x \sqsupseteq u$ for all $u \notin \{x, y\}$, and either $u \sqsupseteq v$ or $u \oplus v$ for distinct $u, v \neq x$. Hence for all distinct $u, v \in V$, either f_{uv} is essentially $(n-3)$ -ary or f_{uv} is essentially $(n-1)$ -ary and is determined by oddsupp. It follows that f has no essentially $(n-2)$ -ary collapse. As $n-2 \geq \max(k, 3)$, Theorem 2.5 implies that f depends on oddsupp, contrary to our assumptions.

Claim 3. *If x, y are distinct and f_{xy} is essentially $(n-1)$ -ary, then every two-variable collapse of f_{xy} is essentially $(n-2)$ -ary.*

For suppose this were not the case. Pick x, y such that f_{xy} is essentially $(n-1)$ -ary and for which some two-variable collapse has an inessential variable. f_{xy} is totally symmetric by the hypotheses of the theorem, and hence is determined by oddsupp by Lemma 2.2(3). This contradicts the previous claim.

Claim 4. *If x, y, z, w are distinct and $x \sqsupseteq y$, then (i) $x \sqsupseteq z$ or $x \circ z$ or $x \sqsupseteq y \sqsupseteq z$, and (ii) $z \sqsupseteq w$ or $z \sqsupseteq x \sqsupseteq w$ or $z \sqsupseteq y \sqsupseteq w$.*

Indeed, assume $x \sqsupseteq y$. Then f_{xyz} and $f_{xy,zw}$ depend on all of their variables by Claim 3. One now argues as in the proof of Lemma 1.1(2, 3).

In the next two claims we shall use $x \diamond y$ to mean $x \sqsupseteq y$ or $x \circ y$.

Claim 5. *If there exist distinct x, y, z such that $x \diamond y$ and $x \diamond z$, then $u \diamond v$ for all $u \neq v$.*

Assume $x \diamond y$ and $x \diamond z$. Using Claim 4 and Lemma 1.1(1) we find that $x \diamond u$ for all $u \neq x$, and either $u \diamond v$ or $u \sqsupseteq x \sqsupseteq v$ for all distinct $u, v \neq x$. Suppose that the present claim is false. Then there exist distinct $u, v \neq x$ such that $u \sqsupseteq x \sqsupseteq v$. Observe that if u, v, w are distinct and different from x , then we cannot have $u \sqsupseteq x \sqsupseteq v$ while $v \diamond w$ (by Claim 4 and Lemma 1.1(1)). Hence $u \sqsupseteq x \sqsupseteq v$ for all distinct $u, v \neq x$. Since $n-1 > k$ it follows that f does not depend on x , which is a contradiction.

Claim 6. *$x \sqsupseteq y$ for all $x \neq y$.*

By the hypothesis of the theorem, there exist $x \neq y$ such that $x \sqsubseteq y$. If there does not exist $z \notin \{x, y\}$ such that $x \diamond z$ or $y \diamond z$, then by Claim 4, $x \sqsubseteq y \sqsubseteq z \sqsubseteq x$ for all $z \notin \{x, y\}$. Choose distinct $z, w \notin \{x, y\}$. $x \sqsubseteq y$ implies $z \sqsubseteq w$ or $z \sqsubseteq x$ or $z \sqsubseteq y$ or $z \sqsubseteq w$ by Claim 4. But $z \sqsubseteq w$ is inconsistent with $x \sqsubseteq y \sqsubseteq z$ by Claim 4, $z \sqsubseteq x$ is inconsistent with $x \sqsubseteq y \sqsubseteq z$ by Lemma 1.1(2), and likewise $z \sqsubseteq y$ is inconsistent with $y \sqsubseteq x \sqsubseteq w$.

Hence with no loss of generality we can choose $z \notin \{x, y\}$ such that $x \diamond z$. Then $u \diamond v$ for all $u \neq v$, by Claim 5. Since at least one edge is labelled by \square , since there cannot exist distinct z, w, u, v with $z \sqsubseteq w$ and $u \circ v$ by Claim 4, and since $n \geq 5$, it follows that all edges are labelled by \square .

Claim 7. For all $x \neq y$, f_{xy} is determined by supp.

To see this, note that $n \geq 5$ and write $V = \{x, y, u, v, z, \bar{w}\}$. Put $f(x, x, u, v, z, \bar{w}) = h(x, u, v, z, \bar{w})$ and $f(x, y, u, u, z, \bar{w}) = g(x, y, u, z, \bar{w})$ where by Claim 6 h and g depend on all of their variables and therefore are totally symmetric. Then

$$\begin{aligned} h(x, u, u, z, \bar{w}) &= g(x, x, u, z, \bar{w}) \\ &= g(x, x, z, u, \bar{w}) \quad \text{as } g \text{ is totally symmetric} \\ &= h(x, z, z, u, \bar{w}). \end{aligned}$$

Since h is totally symmetric, it follows that h is determined by supp.

Now we can prove the theorem. It must be shown that f is totally symmetric. Fix $x \neq y$ and write $V = \{x, y, \bar{z}\}$. It is enough to show that for all $a, b \in A$ and $\bar{c} \in A^{n-2}$, $f(a, b, \bar{c}) = f(b, a, \bar{c})$. This is certainly true if either $a = b$ or $c_i = c_j$ for some $i \neq j$ (using the fact that $f_{z_i z_j}$ is totally symmetric in the latter case). If neither holds, then $n = k + 2$ and there exist $i \neq j$ such that $a = c_i$ and $b = c_j$. By symmetry we may assume that $i = 1$ and $j = 2$. Let $\bar{e} = (c_3, \dots, c_{n-2})$. Let $z = z_1$, $w = z_2$, and write $f(x, y, x, w, \bar{u}) = h(x, y, w, \bar{u})$ and $f(x, y, y, w, \bar{u}) = g(x, y, w, \bar{u})$, where h and g are determined by supp by Claim 7. Then

$$\begin{aligned} f(a, b, \bar{c}) &= f(a, b, a, b, \bar{e}) \\ &= h(a, b, b, \bar{e}) \\ &= h(a, a, b, \bar{e}) \quad \text{since } h \text{ is determined by supp} \\ &= f(a, b, a, a, \bar{e}) \\ &= f(b, a, a, a, \bar{e}) \quad \text{since } f_{zw} \text{ is totally symmetric} \\ &= g(b, a, a, \bar{e}) \\ &= g(b, a, b, \bar{e}) \quad \text{since } g \text{ is determined by supp} \\ &= f(b, a, a, b, \bar{e}) \\ &= f(b, a, \bar{c}) \end{aligned}$$

as desired. \square

Corollary 2.7. *Under the assumptions stated at the beginning of this section:*

1. *If $n \geq \max(k, 3) + 2$ and f is not totally symmetric, then there exist distinct x, y such that f_{xy} depends on all of its variables and is not totally symmetric.*
2. *If every essentially m -ary collapse of f is totally symmetric for all $m < \max(k, 3) + 2$, then f is totally symmetric.*

Corollary 2.7(2) is a slight improvement of Theorem 2.12 in [2]. The bound $\max(k, 3) + 2$ in Theorem 2.6 and Corollary 2.7 is tight, as witnessed by the 4-ary operation $[x \wedge (y \vee z \vee w)] \vee [y \wedge z \wedge w]$ in the two-element lattice, and the following family of examples.

Example. Let A be a k -element set, $k \geq 2$, and define $f : A^{k+1} \rightarrow \{0, 1\}$ by

$$f(x_0, \dots, x_k) = \begin{cases} 1 & \text{if } x_1, \dots, x_k \text{ are distinct and } x_0 \neq x_1, \\ 0 & \text{otherwise.} \end{cases}$$

f is not totally symmetric, but every two-variable collapse of f is determined by supp.

3. Characterizing $S(A)$

In this and the next section, Odd denotes the set of all odd positive integers.

Let A be a finite algebra. Following [10, 5] we let

$$S(A) = \{n \in \mathbb{N} : n \neq 1 \text{ and } p_n(A) > 0, \text{ or } n = 1 \text{ and } p_1(A) > 1\}.$$

Lemma 3.1. *Let A be a k -element algebra and $S = S(A)$. Suppose $n > \max(k, 3)$ is such that $n \in S$ while $n - 1 \notin S$. Then:*

1. $0 \notin S$.
2. $[n - 1, \infty) \cap S \subseteq \text{Odd}$. (In particular, n is odd.)
3. $\text{Odd} \cap [3, n] \subseteq S$.

We shall see in the next section that the last item can be strengthened to $\text{Odd} \setminus \{1\} \subseteq S$.

As an immediate consequence of the above lemma we have

Corollary 3.2. *Let A be a finite algebra. Then $S(A)$ is equal modulo a finite set to one of \emptyset , \mathbb{N} , or Odd .*

Proof of Lemma 3.1. The verification of the following claim is left to the reader.

Claim. *Suppose $m > k$ and $g : A^m \rightarrow A$ is determined by oddsupp . Then for any $c \in A$ the operation $h : A^{m-1} \rightarrow A$ defined by $h(x_1, \dots, x_{m-1}) = g(x_1, \dots, x_{m-1}, c)$ is totally*

symmetric and has the same range as g . Hence h depends on all of its variables if g is nonconstant.

To prove item 1, suppose instead that $0 \in S$. Choose $f(x) \in \text{Clo } A$ such that $f(x)$ is constant, say $f(x) = c$. As $n \in S$ we can pick $g \in E_n(A)$. By Corollary 2.3, g is determined by oddsupp . Let

$$\begin{aligned} h(x_1, \dots, x_{n-1}) &= g(x_1, \dots, x_{n-1}, f(x_1)) \\ &= g(x_1, \dots, x_{n-1}, c). \end{aligned}$$

Then $h \in \text{Clo } A$ and h depends on all of its variables by the claim, which contradicts the fact that $n-1 \notin S$.

Next suppose that $m \in [n-1, \infty) \cap S$. Thus $m \geq n$. Pick $f \in E_m(A)$. No collapse of f is essentially $(n-1)$ -ary, so f is determined by oddsupp and $m \equiv n \pmod{2}$ by Theorem 2.5. Furthermore, no collapse of f is constant (as $0 \notin S$) so repeated applications of Lemma 2.2(4) show that f has an essentially r -ary collapse for all $r < m$ such that $r \equiv m \pmod{2}$. Applying these observations to some $f \in E_n(A)$ (which must exist as $n \in S$), and observing that f has no essentially 0-ary collapse, we find that n is odd. These remarks establish items 2 and 3. \square

Urbanik [10] characterized those subsets $S \subseteq \mathbb{N}$ satisfying $S \cap \{0, 1\} = \emptyset$ (the ‘idempotent case’) which are equal to $S(A)$ for some finite algebra A , and showed moreover that each such S occurs as $S(A)$ for some finite algebra A of finite type. We now do the same thing for the nonidempotent case. (This solves the first part of Problem 3 in [5].)

Theorem 3.3. *Suppose A is a finite algebra, $S = S(A)$, and $S \cap \{0, 1\} \neq \emptyset$. Then one of the following conditions holds:*

1. S is finite (and $S \cap \{0, 1\} \neq \emptyset$);
2. $S = \text{Odd} \cup X$ where X is finite and $0 \notin X$.
3. $S = \mathbb{N} \setminus X$ where X is finite (and $\{0, 1\} \not\subseteq X$).

Moreover, each set in the list occurs as $S(A)$ for some finite algebra A of finite type.

Proof. The necessity of the conditions follows from Corollary 3.2 and Lemma 3.1(1, 3). An outline of the proof of sufficiency follows.

Claim 1. *Suppose A is a finite algebra of finite type. Then for every finite set $X \subseteq \mathbb{N} \setminus \{0\}$ there exists a finite algebra B of finite type satisfying $S(B) = S(A) \cup X \cup \{1\}$.*

To prove this, let $N = \max(X)$, let C be an N -element set disjoint from A , and let $B = A \cup C \cup \{\infty\}$ where we assume $\infty \notin A \cup C$. Let \mathcal{L} be a language indexing the

fundamental operations of A . Fix $a_0 \in A$ and define $e: B \rightarrow A$ by $e(x) = x$ if $x \in A$, $e(x) = a_0$ otherwise. For each $f \in \mathcal{L}$ of arity m define

$$f^B(x_1, \dots, x_m) = f^A(e(x_1), \dots, e(x_m)).$$

For each $r \in X$ define

$$g_r(x_1, \dots, x_r) = \begin{cases} \infty & \text{if } x_1, \dots, x_r \text{ are distinct elements of } C \\ e(x_1) & \text{otherwise.} \end{cases}$$

Let $B = \langle B; f^B(f \in \mathcal{L}), g_r(r \in X), e \rangle$. Note that if t is any term in the language of B , $g_r(t_1, \dots, t_r)$ is a proper subterm of t , and t' is obtained from t by replacing one occurrence of $g_r(t_1, \dots, t_r)$ by t_1 , then t and t' define the same operation in B . Similarly, if $t = g_r(t_1, \dots, t_r)$ and some t_i is not a variable, then t and $e(t_i)$ define the same operation in B . Thus every term operation of B has one of the following forms: (1) $f(e(x_1), \dots, e(x_m))$ for some $f \in \text{Clo } A$; (2) $g_r(x_{i_1}, \dots, x_{i_r})$. The rest of the proof that $S(B) = S(A) \cup X \cup \{1\}$ is left to the reader.

Claim 2. Suppose A is a finite algebra of finite type with no 0-ary fundamental operations and having an element $0 \in A$ such that (i) $\{0\}$ is the range of some term operation; (ii) 0 is an absorbing element for A , i.e., if f is an m -ary fundamental operation of A and $\bar{a} \in A^m$, then $0 \in \{a_1, \dots, a_m\}$ implies $f(\bar{a}) = 0$; and (iii) if $t(x)$ is a unary term other than a variable, then $t^A(x)$ is constant.

Then for every finite subset $X \subseteq \mathbb{N}$ there exists a finite algebra B of finite type such that $S(B) = S(A) \cup X$.

To prove this, first observe that the hypotheses imply $0 \in S(A)$, so we can assume that $0 \notin X$. Also, every constant term operation of A has range $\{0\}$, and if t is any term such that t^A is nonconstant, then t^A depends on all of the variables which occur in t .

Let N , C , ∞ and B be as in the proof of Claim 1. Let \mathcal{L} be the language of A . For each $f \in \mathcal{L}$ of arity m define

$$f^B(x_1, \dots, x_m) = \begin{cases} f^A(x_1, \dots, x_m) & \text{if } x_1, \dots, x_m \in A \\ 0 & \text{otherwise.} \end{cases}$$

For each $r \in X$ define

$$g_r(x_1, \dots, x_r) = \begin{cases} \infty & \text{if } x_1, \dots, x_r \text{ are distinct elements of } C \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \langle B; f^B(f \in \mathcal{L}), g_r(r \in X) \rangle$.

The hypotheses and remarks at the beginning of the proof of this claim imply that, for any \mathcal{L} -term t : (1) t^A and t^B have the same essential arity; (2) t^B is not essentially unary unless t is a variable. Further analysis of the terms reveals that every term operation of B has the form t^B for some \mathcal{L} -term t , or $g_r(x_{i_1}, \dots, x_{i_r})$. The rest of the proof that $S(B) = S(A) \cup X$ is left to the reader.

We now prove that each set S of types 1–3 in the statement of the theorem is representable as $S(A)$ for some finite algebra of finite type. For sets of types 1 and 2 it suffices, by Claims 1 and 2, to observe that (a) \emptyset and Odd are representable by finite algebras of finite type (for Odd see Lemma 4.3), and (b) $\{0\}$ is representable by an algebra satisfying the hypotheses of Claim 2. For sets of type 3, the following two claims will suffice.

Claim 3. *For each $m \geq 2$ there exists a finite algebra A of finite type such that $S(A) = [m, \infty)$.*

Claim 4. *For each $m \geq 2$ there exists a finite algebra A of finite type satisfying the hypotheses of Claim 2 and such that $S(A) = \{0\} \cup [m, \infty)$.*

To prove Claim 3, let $A = \langle A; f \rangle$ where A is a set of cardinality $m + 1$ and

$$f(x_1, \dots, x_m) = \begin{cases} y & \text{if } \{x_1, \dots, x_m, y\} = A, \\ x_1 & \text{if } x_1, \dots, x_m \text{ are not pairwise distinct.} \end{cases}$$

It is easy to check that $S(A) \subseteq [m, \infty)$. For the opposite inclusion let $y_1, \dots, y_{m-2}, x_0, x_1, x_2, \dots$, be distinct variables and consider the term operations

$$f(f(f(\dots f(f(x_0, x_1, \bar{y}), x_2, \bar{y}) \dots), x_{r-1}, \bar{y}), x_r, \bar{y}).$$

The proof of Claim 4 is identical to the proof of Claim 3 except that an element $0 \in A$ is chosen and the operation f is defined so that $f(x_1, \dots, x_m) = 0$ if x_1, \dots, x_m are not pairwise distinct. The details are left to the reader.

This concludes the proof of the theorem. \square

4. The eventual behavior of $p_n(A)$

Recall that if $f : A^n \rightarrow A$ and $\sigma \in S_n$, then $f^\sigma : A^n \rightarrow A$ is defined by $f^\sigma(\bar{x}) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Let $G(f) = \{\sigma \in S_n : f^\sigma = f\}$, a subgroup of S_n . Thus f is totally symmetric if and only if $G(f) = S_n$. Interestingly, if $|A|$ is fixed and n is large then not every subgroup of S_n arises in this way.

Lemma 4.1 (Kisielewicz). *Suppose f is an n -ary operation on a k -element set, with $n > k$. Then $G(f) \neq A_n$.*

Proof. Say $f : C^n \rightarrow C$, where $|C| = k$. Suppose $A_n \subseteq G(f)$. We shall prove that the transposition $(1\ 2)$ is in $G(f)$. Let $(c_1, \dots, c_n) \in C^n$ be given. We wish to prove that $f^{(1\ 2)}(\bar{c}) = f(\bar{c})$. If $c_1 = c_2$ then this is obviously true. Suppose there exist i, j with $3 \leq i < j \leq n$ such that $c_i = c_j$. Then $f^{(1\ 2)}(\bar{c}) = f^{(1\ 2)(i\ j)}(\bar{c}) = f(\bar{c})$, the last

equation holding since $(1\ 2)(i\ j) \in A_n \subseteq G(f)$. Finally, suppose $c_1 \neq c_2$ and $c_i \neq c_j$ whenever $3 \leq i < j \leq n$. As $n > k$ we have $\{c_1, c_2\} \cap \{c_3, \dots, c_n\} \neq \emptyset$. By symmetry we may assume that $c_1 = c_3$. Then $f^{(1\ 2)}(\bar{c}) = f^{(1\ 2\ 3)}(\bar{c}) = f(\bar{c})$ as $(1\ 2\ 3) \in A_n$. Thus $f^{(1\ 2)} = f$, so $(1\ 2) \in G(f)$ and hence $G(f) = S_n$. \square

Theorem 4.2. *Suppose C is a clone on a k -element set. Then either the sequence $\langle p_n(C) \rangle_{n \in \omega}$ is bounded or $p_n(C) \geq n$ for all $n > \max(k, 4)$.*

Remark. There exist finite semigroups S with $p_n(S) = n$ for all $n \geq 0$ (see [9], Lemma 2).

Proof of Theorem 4.2. First observe that if $f \in E_n(C)$ and $[S_n : G(f)] = i$ then the orbit of f under the action of S_n on $E_n(C)$ has i distinct elements, so $p_n(C) \geq i$. Now we consider cases.

Case 1: For arbitrarily large n there exists $f_n \in E_n(C)$ where f_n is not totally symmetric.

Then by Corollary 2.7(1) there exists such $f_n \in E_n(C)$ for each n satisfying $n > \max(k, 3)$. $G(f_n) \neq S_n$ as f_n is not totally symmetric, and $G(f_n) \neq A_n$ by Lemma 4.1. If $n \neq 4$ this implies $[S_n : G(f_n)] \geq n$. Hence $p_n(C) \geq n$ for all $n > \max(k, 4)$.

Case 2: There exists N such that for all $n \geq N$, if $f \in E_n(C)$ then f is totally symmetric.

We may assume that $N > \max(k, 3)$. By Theorem 2.6, if $n > N$ then each $f \in E_n(C)$ is determined by either *supp* or *oddsupp*. Thus by the proof of Lemma 2.5 in [2], $p_n(C) \leq k^{2^k}$ for all $n > N$. \square

Next we refute the following conjecture of Berman [1], which also appears as Problem 5 in [5]

Conjecture. If A is a finite algebra of finite type, then the p_n -sequence of A is either eventually strictly increasing or bounded above by a constant.

Recall that a *Boolean group* is an abelian group satisfying $x + x = 0$, and that for every $d \geq 0$ there exists a Boolean group of cardinality 2^d . Suppose $\langle B; +, 0 \rangle$ is a Boolean group having more than one element; define $d(x, y, z) = x + y + z$ and for each $a \in B$ define $f_a(x) = x + a$. Let $A = \langle B; d, f_a (a \in B) \rangle$. Then $p_{2n}(A) = 0$ and $p_{2n+1}(A) = |B|$ for all $n \geq 0$. This explains the following (well-known) fact.

Lemma 4.3. *For arbitrarily large N there exists a finite algebra of finite type whose p_n -sequence is $\langle 0, N, 0, N, 0, N, \dots \rangle$.*

Next we show

Lemma 4.4. *Let A be a finite algebra. There exists an algebra B with $|B| = 2 \cdot |A|$, such that $p_n(B) = p_n(A) + n$ for all $n < \omega$. Moreover, B can be chosen to be of finite type if A is of finite type.*

Proof. With no loss of generality we may assume that A has no 0-ary fundamental operations, and has a unary fundamental operation p satisfying $p(x) = x$. Let \mathcal{L} be a language indexing the fundamental operations of A , let m be a new binary operation symbol, and let $\mathcal{L}' = \mathcal{L} \cup \{m\}$. Expand A to an \mathcal{L}' -algebra A' by defining $m^{A'}(x, y) = x$ for all $x, y \in A$. Let $S = \langle \{0, 1\}; m^S \rangle$ be the two-element meet semilattice, and expand S to an \mathcal{L}' -algebra S' by defining $f^{S'}(x_1, \dots, x_n) = 0$ for each n -ary operation symbol $f \in \mathcal{L}$. Now let $B = A' \times S'$.

If t is any \mathcal{L}' -term mentioning precisely the variables x_1, \dots, x_n , then $t^C = (t^{A'}, t^{S'})$ has the form $(s^A, 0)$ or $(x_i, x_1 \wedge \dots \wedge x_n)$ where s ranges over \mathcal{L} -terms and $1 \leq i \leq n$. Moreover, the reader can check that each possible pair of terms of the above form arises from some \mathcal{L}' -term t . Thus $p_n(B) = p_n(A) + n$ for all $n \geq 0$. \square

Now to refute Berman's conjecture, let A be a finite algebra of finite type whose p_n -sequence is $\langle 0, N, 0, N, \dots \rangle$. Let B be the algebra (also finite and of finite type) obtained from A by the previous lemma. The p_n -sequence of B is $\langle 0, N+1, 2, N+3, 4, N+5, \dots \rangle$, which is neither bounded nor eventually strictly increasing.

We remark that this counterexample has the property that $p_n(B) - p_{n+1}(B)$ is bounded above.

Problem. Does there exist a finite algebra A (of finite type) such that $\langle p_n(A) - p_{n+1}(A) \rangle_{n < \omega}$ is not bounded above?

The final topic of this paper is the eventual behavior of the p_n -sequences of algebras which satisfy the hypotheses of Lemma 3.1.

Let A be a finite set with $|A| > 1$. Suppose we are given the following data: (1) a nonempty collection Σ of nonconstant maps $f : A \rightarrow A$ satisfying $f \circ f = f$ and $\ker(f) = \ker(f')$ for all $f, f' \in \Sigma$; (2) a designated member $e \in \Sigma$, with image $U := e(A)$; (3) a binary operation $+$ on U and element $0 \in U$ making $\langle U; +, 0 \rangle$ a Boolean group; and (4) a subgroup H of $\langle U, + \rangle$. For $n \in \text{Odd}$, $f \in \Sigma$ and $a \in H$ define $F_{f,n,a} : A^n \rightarrow A$ by

$$F_{f,n,a}(x_1, \dots, x_n) = f(e(x_1) + \dots + e(x_n) + a).$$

Also let $d(x, y, z) = F_{e,3,0}(x, y, z) = e(x) + e(y) + e(z)$. Finally, let

$$\begin{aligned} C(\Sigma, U, +, H) &= \{F_{f,n,a} : f \in \Sigma, n \in \text{Odd}, a \in H\} \cup \{\text{id}_A\} \\ C &= \langle A; d, f \ (f \in \Sigma), e(x) + a \ (a \in H) \rangle. \end{aligned}$$

Observe that $e(u) = u$ for all $u \in U$, and that $f \circ f' = f$ for all $f, f' \in \Sigma$. The reader can check that:

- $C(\Sigma, U, +, H) = \bigcup_{m \geq 0} E_m(C)$.
- The p_n -sequence of C is $(0, x, 0, N, 0, N, \dots, 0, N, \dots)$, where $N = |\Sigma| \cdot |H|$ and $x = N$ if $U = A$, $x = N + 1$ otherwise.

The following theorem improves Lemma 3.1.

Theorem 4.5. *Let A be a finite algebra and $k = |A|$. Suppose $n > \max(k, 3)$ is such that $n \in S(A)$ while $n - 1 \notin S(A)$. Then there exist $\Sigma, U, +, H$ as above such that*

1. $C(\Sigma, U, +, H) \subseteq \text{Clo } A$.
2. For all $m \geq n$, $E_m(A) \subseteq C(\Sigma, U, +, H)$. Hence for $m \geq n$ we have

$$p_m(A) = \begin{cases} N & \text{if } m \in \text{Odd}, \\ 0 & \text{otherwise} \end{cases}$$

and $p_m(A) \geq N$ for all odd $m < n$, where $N = |\Sigma| \cdot |H|$.

Proof. Observe first that if $m \geq n$ and $g \in E_m(A)$, then m is odd, g is determined by oddsupp , and every collapse of g is nonconstant (by Theorem 2.5 and Lemma 3.1).

Define Γ to be the set of all $f \in E_r(A)$, $r \in \text{Odd}$, such that for some $m = r + 2t \geq n$ ($t \geq 0$) and some $F \in E_m(A)$, $f(\bar{x}) = F(\bar{x}, u, u, \dots, u, u)$. Note that every member of Γ is determined by oddsupp .

Claim 1. *Suppose $f(x_1, \dots, x_r) \in \Gamma$ and $g(x_1, \dots, x_s) \in \text{Clo } A$ where g is totally symmetric. Then the operation $f(g(x_1, \dots, x_s), x_{s+1}, \dots, x_{s+r-1})$ is also in Γ .*

To see this, choose $F(x_1, \dots, x_r, u_1, \dots, u_{2t}) \in E_m(A)$ where $m = r + 2t \geq n$ and $f(\bar{x}) = F(\bar{x}, u, \dots, u)$. Note that m is odd. Let

$$t(y_1, \dots, y_s, x_2, \dots, x_r) = f(g(\bar{y}), \bar{x})$$

and

$$T(y_1, \dots, y_s, x_2, \dots, x_r, u_1, \dots, u_{2t}) = F(g(\bar{y}), \bar{x}, \bar{u}).$$

Pick any $c \in \text{range}(g)$. By the claim in the proof of Lemma 3.1, $F(c, \bar{x}, \bar{u})$ depends on all of its variables. Hence $T(\bar{y}, \bar{x}, \bar{u})$ depends on \bar{x} and \bar{u} at least. Suppose T does not depend on any y_i . Then the essential arity of T is $r - 1 + 2t = m - 1$. Obviously, $m \neq n$ as $p_{n-1}(A) = 0$, so $m - 1 \geq n$ and $p_{m-1}(A) > 0$. But $m - 1$ is even, which contradicts the observation at the beginning of the proof of the theorem. Therefore, T depends on at least one y_i , and therefore depends on all of its variables as g is totally symmetric. Hence $T \in E_q(A)$, where $q = s + r - 1 + 2t \geq n$. As $t(\bar{y}, \bar{x}) = T(\bar{y}, \bar{x}, u, \dots, u, u)$ it follows that $t \in \Gamma$ as required.

Claim 2. Suppose $f(x_1, \dots, x_r) \in \Gamma$. Let $\alpha(x) = f(x, x, \dots, x)$. Then (i) f commutes with itself, and (ii) $f(\alpha(x_1), \dots, \alpha(x_r)) = \alpha(f(x_1, \dots, x_r))$.

Indeed, repeated applications of Claim 1 show that

$$f(f(x_1^1, \dots, x_r^1), \dots, f(x_1^r, \dots, x_r^r))$$

is in Γ , hence is totally symmetric. This proves (i); (ii) follows from (i) easily.

Claim 3. There exists $e : A \rightarrow A$ satisfying $e^2 = e$, and there exists a binary operation $+$ on $U := e(A)$ making $\langle U; + \rangle$ a Boolean group, such that both $e(x)$ and $d(x, y, z) := e(x) + e(y) + e(z)$ are in Γ .

To prove this, start with any $g \in E_n(A)$. Recall that n is odd and $n > 3$. Define $f(x, y, z) = g(x, y, z, u, \dots, u, u) \in \Gamma$ and $\alpha(x) = f(x, x, x)$. By the finiteness of $|A|$ there exists $k > 0$ such that $\alpha^{2k+1} = \alpha^k$. Define $e = \alpha^{k+1}$ and $d(x, y, z) = \alpha^k(f(x, y, z))$. Then by construction we have $e^2 = e$, $d(x, x, x) = e(x)$, and $ed(x, y, z) = d(x, y, z)$. Repeated applications of Claim 1 also show that $d, e \in \Gamma$. Let $U = e(A)$. Pick any element $0 \in U$ and for $x, y \in U$ define $x + y = d(x, 0, y) \in U$.

Let \hat{d} be the restriction of d to U . Note that $\hat{d}(x, y, y) = \hat{d}(y, y, x) = \hat{d}(y, x, y) = \hat{d}(x, x, x) = e(x) = x$ for all $x, y \in U$. This proves that \hat{d} is a Mal'cev operation on U , and also that $x + x = 0$ for all $x \in U$. Moreover, \hat{d} commutes with itself by Claim 2(i). By Lemma 5.6 of [4] it follows that $\langle U; + \rangle$ is a Boolean group and $\hat{d}(x, y, z) = x + y + z$ for all $x, y, z \in U$. Then combining Claim 2(ii) with an observation above we have that $d(x, y, z) = ed(x, y, z) = d(e(x), e(y), e(z)) = e(x) + e(y) + e(z)$ for all $x, y, z \in A$, as required.

Now fix some choice of $e, U, +$ and d witnessing Claim 3, and define

$$\Sigma = \{f \in E_1(A) : f^2 = f \text{ and } \ker(f) = \ker(e)\},$$

$$H = \{a \in U : e(x) + a \text{ is a term operation of } A\}.$$

Clearly $C(\Sigma, U, +, H) \subseteq \text{Clo } A$, which proves the first item in the statement of the theorem. To prove the second item, several more claims are needed.

Claim 4. Suppose $f(x_1, \dots, x_r) \in \Gamma$, $r \geq 1$. Then $f(\bar{a}) = f(e(a_1), a_2, \dots, a_r)$ for all \bar{a} .

This can be seen as follows. Choose $F \in E_m(A)$ such that $m \geq n$ and $f(\bar{x}) = F(\bar{x}, u, \dots, u)$. It suffices to prove the claim with F and m in place of f and r . Let $F^*(x_1, \dots, x_m) = F(e(x_1), x_2, \dots, x_m)$. Then $F^* \in \Gamma$ by Claim 1, hence $F^*(y, y, x_3, \dots, x_m)$ as well as $F(y, y, x_3, \dots, x_m)$ do not depend on y . Now let $\bar{a} \in A^m$ be given. It must

be shown that $F(\bar{a}) = F^*(\bar{a})$. As $m > k$ there exist $i < j$ such that $a_i = a_j$. By the total symmetry of F and F^* we may assume that $(i, j) = (1, 2)$. Then

$$\begin{aligned} F^*(a_1, a_1, a_3, \dots, a_m) &= F^*(0, 0, a_3, \dots, a_m) \\ &= F(e(0), 0, a_3, \dots, a_m) \\ &= F(0, 0, a_3, \dots, a_m) \quad \text{as } e(0) = 0 \\ &= F(a_1, a_1, a_3, \dots, a_m). \end{aligned}$$

Claim 5. Suppose $f(x_1, \dots, x_r) \in \Gamma$. Let $\alpha(x) = f(x, \dots, x)$. Then $f(\bar{x}) = \alpha(e(x_1) + \dots + e(x_r))$ for all $x_i \in A$.

To prove this, note that $\alpha(x) = f(x, x, \dots, x) = f(x, 0, \dots, 0)$ (since f is determined by oddsupp) and let

$$T(x, y_2, \dots, y_r, z_2, \dots, z_r) = f(e(x) + e(y_2) + \dots + e(y_r), z_2, \dots, z_r).$$

By Claim 1, T is totally symmetric. Hence

$$\begin{aligned} \alpha(e(x) + e(y_2) + \dots + e(y_r)) &= f(e(x) + e(y_2) + \dots + e(y_r), 0, \dots, 0) \\ &= T(x, y_2, \dots, y_r, 0, \dots, 0) \\ &= T(x, 0, \dots, 0, y_2, \dots, y_r) \\ &= f(e(x) + 0 + \dots + 0, y_2, \dots, y_r) \\ &= f(x, y_2, \dots, y_r), \end{aligned}$$

where the last line is true by Claim 4.

Claim 6. For each unary term operation $\alpha(x)$ of A there exists $a \in H$ such that $e\alpha(x) = e(x) + a$ for all $x \in A$.

To prove this, let $p(x, y, z) = d(\alpha(x), y, z) = e\alpha(x) + e(y) + e(z)$. $p \in \Gamma$ by Claim 1, so $p(x, x, z)$ does not depend on x , i.e., $e\alpha(x) + e(x)$ is constant. Thus $e\alpha(x) = e(x) + a$ for some $a \in U$. $e\alpha(x)$ is a term operation of A , so $a \in H$.

Claim 7. If $\alpha(x) \in \Gamma$ and $a \in H$ satisfy $e\alpha(x) = e(x) + a$, then $\alpha^2 \in \Sigma$ and $\alpha(x) = \alpha^2(e(x) + a)$ for all $x \in A$.

Indeed, we have $e\alpha^2 = e$ by assumption and $\alpha e = \alpha$ by Claim 4. These imply $\ker(\alpha) = \ker(e) = \ker(\alpha^2)$ and $\alpha = \alpha^3 = \alpha^2 e\alpha$. Thus $\alpha^2 \in \Sigma$ and $\alpha(x) = \alpha^2(e\alpha(x)) = \alpha^2(e(x) + a)$ by Claim 6, as desired.

Now we can finish the proof of the theorem. Suppose $m \geq n$ and $f \in E_m(A)$. Let $\alpha(x) = f(x, \dots, x)$ and choose $a \in H$ witnessing Claim 6. Let $\beta = \alpha^2 \in \Sigma$. Then $f = F_{\beta, m, a} \in C(\Sigma, U, +, H)$ by Claims 5 and 7 and the fact that $e(e(x_1) + \dots + e(x_m)) = e(x_1) + \dots + e(x_m)$. \square

The same methods show

Corollary 4.6. *Let $A, \Sigma, e, U, +, H$ be as in the statement of Theorem 4.5. Let $\theta = \ker(e)$. Then:*

1. $\theta \in \text{Con } A$
2. *For each $f(x_1, \dots, x_m) \in \text{Clo } A$ there exists $I \subseteq \{1, \dots, m\}$ and $a \in H$ such that*
 - (i) $|I|$ *is odd, and* (ii) $f(x_1, \dots, x_m) \stackrel{\theta}{=} (\sum_{i \in I} e(x_i)) + a$ *for all $\bar{x} \in A^m$.*
3. *If $f \in E_m(A)$ is totally symmetric, then m is odd and $I = \{1, \dots, m\}$ in the previous item.*

Proof. We first prove item 2. Suppose $f(x_1, \dots, x_m) \in \text{Clo } A$. Note that $ef(\bar{x})$ is not constant. We may assume with no loss of generality that $ef(\bar{x})$ depends on all of its variables. Let $P(\bar{x}, y_1, \dots, y_{n-1}) = ef(\bar{x}) + \sum_i e(y_i)$. Clearly $P \in \text{Clo } A$ and P depends on all of its variables. Hence P is totally symmetric (as its essential arity is at least n), and therefore $ef(\bar{x})$ is totally symmetric. Thus $ef(\bar{x}) = e(ef(\bar{x})) \in \Gamma$ by Claim 1 in the proof of Theorem 4.5. Claims 5–7 yield $ef(\bar{x}) = \sum_i e(x_i) + a$ for some $a \in H$, which proves item 2. Item 1 follows immediately from item 2. To prove item 3, note that $ef(x_1, \dots, x_m) \in \Gamma$ by Claim 1 in the proof of Theorem 4.5. \square

Theorem 4.5 and its corollary are small steps toward solving Problem 6 in [5]:

Problem. Characterize the p_n -sequences of finite totally symmetric algebras.

Acknowledgements

I am grateful to Joel Berman and Andrzej Kisielewicz for their valuable comments.

References

- [1] J. Berman, Lecture notes: free spectra of finite algebras, manuscript, 1986.
- [2] J. Berman and A. Kisielewicz, On the number of operations in a clone, *Proc. Amer. Math. Soc.* 122 (1994) 359–369.
- [3] S. Fajtlowicz, On algebraic operations in binary algebras, *Colloq. Math.* 21 (1970) 23–26.
- [4] R. Freese and R. McKenzie, Commutator theory for congruence modular varieties, *London Math. Soc. Lecture Note Series*, Vol. 125 (Cambridge Univ. Press, Cambridge, 1987).
- [5] G. Grätzer and A. Kisielewicz, A survey of some open problems on p_n -sequences and free spectra of algebras and varieties, in *Universal Algebra and Quasigroup Theory* (Heldermann, Berlin, 1992) 57–88.
- [6] G. Grätzer, J. Płonka and A. Sekanina, On the number of polynomials of a universal algebra I, *Colloq. Math.* 22 (1970) 9–11.
- [7] A. Kisielewicz, Characterization of p_n -sequences for nonidempotent algebras, *J. Algebra* 108 (1987) 102–115.
- [8] A. Kisielewicz, Ternary clones: a problem of Fajtlowicz, *Houston J. Math.* 14 (1988) 515–527.
- [9] J. Płonka, On algebras with n distinct n -ary operations, *Algebra Universalis* 1 (1971) 73–79.
- [10] K. Urbanik, On algebraic operations in idempotent algebras, *Colloq. Math.* 13 (1965) 129–157.