PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

M_n AS A 0, 1-SUBLATTICE OF Con A DOES NOT FORCE THE TERM CONDITION

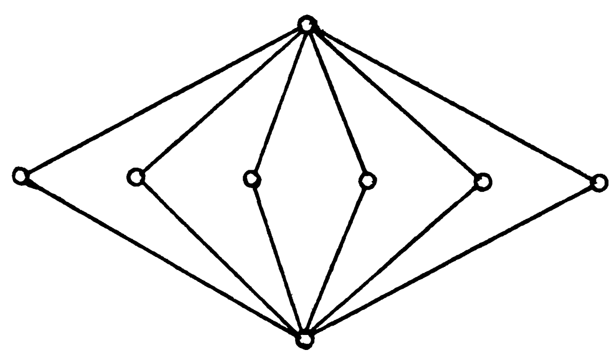
ROSS WILLARD

(Communicated by Donald S. Passman)

ABSTRACT. For every $n \geq 3$ there exists a finite nonabelian algebra whose congruence lattice has M_n as a 0, 1-sublattice. This answers a question of R. McKenzie and D. Hobby.

DEFINITION 0.1. Suppose L and L_1 are bounded lattices. A *copy* of L (in L_1) is any sublattice L' of L_1 which is isomorphic to L. L' is a 0, 1-*copy* of L if it includes the least and greatest elements of L_1; in this case L' is also called a 0, 1-*sublattice* of L_1.

DEFINITION 0.2. For $n \geq 1$, M_n is the finite lattice of height 2 having exactly n atoms. For example, M_6 is



Suppose G is a group and $N(G)$ is its lattice of normal subgroups. There is a trivial proof, using the commutator operation on normal subgroups, that if $N(G)$ has a 0,1-copy of M_3 then G is abelian. This same proof extends, via the general commutator theory of universal algebra, to any algebra $\mathbf{A}$ belonging to a variety whose congruence lattices satisfy the modular law. Here $N(G)$ is replaced by Con $\mathbf{A}$ (the lattice of congruence relations of $\mathbf{A}$), while 'abelian' means the following 'term condition':

DEFINITION 0.3. An algebra $\mathbf{A}$ is *abelian* if for every $n \geq 1$, every $(n+1)$-ary term $t(x, y_1, \ldots, y_n)$ in the language of $\mathbf{A}$, and all a, b, $c_1, \ldots, c_n$, $d_1, \ldots, d_n \in A$,

$$t^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(a, \vec{d}) \quad \text{iff} \quad t^{\mathbf{A}}(b, \vec{c}) = t^{\mathbf{A}}(b, \vec{d}).$$

In their forthcoming book on tame congruence theory [**1**], R. McKenzie and D. Hobby ask whether the above phenomenon—Con $\mathbf{A}$ having M_3 as a 0, 1-sublattice

Received by the editors September 28, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 08A30; Secondary 06B99, 08A40.

Key words and phrases. Congruence, lattice, M_n, abelian, term condition.

forcing **A** to be abelian—holds for all *finite* algebras (without assuming congruence modularity); and if not, whether there is any $n > 3$ for which having M_n as a 0, 1-sublattice suffices. The purpose of this paper is to note that the answer to the first question is positive (without requiring finiteness) in case **A** belongs to a locally finite variety which 'omits type 1', while the answers to both questions are negative in general.

1. In this section I prove the positive answer for algebras belonging to locally finite varieties which omit type 1. Definition 1.1 is taken from [**2**], and Lemma 1.4 is a slight improvement of Lemma 4.153 found there.

DEFINITION 1.1. Suppose α, β, δ are congruences of an algebra **A**.

(1) α *centralizes* β *modulo* δ if, for every $n \geq 1$, every term $t(x, y_1, \ldots, y_n)$ in the language of **A**, and all $\langle a, b\rangle \in \alpha$, $\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle \in \beta$,

$$t^{\mathbf{A}}(a, \vec{c}) \overset{\delta}{\equiv} t^{\mathbf{A}}(a, \vec{d}) \quad \text{iff} \quad t^{\mathbf{A}}(b, \vec{c}) \overset{\delta}{\equiv} t^{\mathbf{A}}(b, \vec{d}).$$

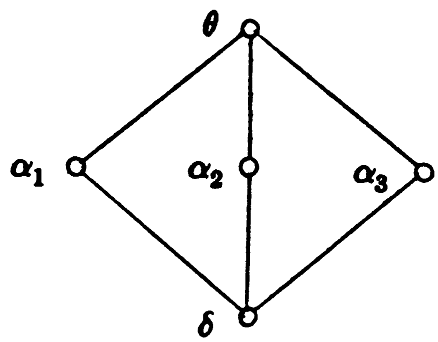
(2) If $\alpha \geq \delta$, then α is *abelian over* δ if α centralizes itself modulo δ. (Thus **A** is abelian iff ∇_A is abelian over Δ_A.)

DEFINITION 1.2. Suppose δ, θ are congruences of **A** with $\theta \geq \delta$.

(1) θ is *solvable over* δ if there exist $\alpha_0, \ldots, \alpha_k \in \operatorname{Con}\mathbf{A}$ such that $\alpha_0 = \delta$, $\alpha_k = \theta$, and α_{i+1} is abelian over α_i for each $i < k$.

(2) θ is *locally solvable over* δ if in every finitely generated subalgebra $\mathbf{B} \leq \mathbf{A}$, $\theta \cap B^2$ is solvable over $\delta \cap B^2$.

LEMMA 1.3. *Suppose* **A** *is an algebra and* Con **A** *has a copy of* M_3, *as in the picture below.*



If $\alpha_2 \circ \alpha_3 \circ \alpha_2 = \theta$, *then* α_1 *centralizes* θ *modulo* α_2, *and hence modulo* δ.

PROOF. Let $t(x, y_1, \ldots, y_n)$ be a term in the language of **A**, and $\langle a, b\rangle \in \alpha_1$, $\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle \in \theta$. Pick c_i', d_i' such that $c_i \overset{\alpha_2}{\equiv} c_i' \overset{\alpha_3}{\equiv} d_i' \overset{\alpha_2}{\equiv} d_i$ for each $i = 1, \ldots, n$. Now suppose $t^{\mathbf{A}}(a, \vec{c}) \overset{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d})$. Then

$$t^{\mathbf{A}}(a, \vec{c}\,') \overset{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{c}) \overset{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d}) \overset{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d}\,') \quad \text{and} \quad t^{\mathbf{A}}(a, \vec{c}\,') \overset{\alpha_3}{\equiv} t^{\mathbf{A}}(a, \vec{d}\,'),$$

so

$$\langle t^{\mathbf{A}}(a, \vec{c}\,'), t^{\mathbf{A}}(a, \vec{d}\,')\rangle \in \alpha_2 \cap \alpha_3 \subseteq \alpha_1.$$

Similarly, $\langle t^{\mathbf{A}}(b, \vec{c}\,'), t^{\mathbf{A}}(b, \vec{d}\,')\rangle \in \alpha_1 \cap \alpha_3 \subseteq \alpha_2$. So $t^{\mathbf{A}}(b, \vec{c}) \overset{\alpha_2}{\equiv} t^{\mathbf{A}}(b, \vec{d})$, which proves that α_1 centralizes θ modulo α_2.

Now α_1 automatically centralizes θ modulo $\alpha_1 \cap \theta = \alpha_1$. It follows that α_1 centralizes θ modulo $\alpha_2 \cap \alpha_1 = \delta$. □

COROLLARY 1.4. *Suppose* Con **A** *has a copy of* M_3 *as in the previous lemma. If for some* $\{i_1, i_2\} = \{2,3\}$ *and* $\{j_1, j_2\} = \{1,3\}$ *the condition*

$$\alpha_{i_1} \circ \alpha_{i_2} \circ \alpha_{i_1} = \alpha_{j_1} \circ \alpha_{j_2} \circ \alpha_{j_1} = \theta$$

is met, then θ *is abelian over* δ.

PROOF. By the previous lemma, α_1 and α_2 both centralize θ modulo δ. It follows that $\alpha_1 \vee \alpha_2 = \theta$ centralizes θ modulo δ. □

The concept of a locally finite variety V omitting type 1 is central to tame congruence theory. One way to define this concept is as follows: V *omits type* 1 if for every $\mathbf{A} \in V$ and every $\theta \in \mathrm{Con}\,\mathbf{A}$ different from Δ_A, there is an $(n+1)$-ary term $t(x, \vec{y})$ in the language of V, and there exist elements $a \overset{\theta}{\equiv} b$, $c_i \overset{\theta}{\equiv} d_i \overset{\theta}{\equiv} e_i$ $(1 \le i \le n)$ of A, such that $t^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(b, \vec{d})$ but $t^{\mathbf{A}}(a, \vec{e}) \ne t^{\mathbf{A}}(b, \vec{e})$. Most interesting locally finite varieties, including those whose congruence lattices satisfy some nontrivial lattice identity, omit type 1. Omitting type 1 is equivalent to the variety satisfying some nontrivial idempotent Mal'cev condition, and also to the permuting of congruences in locally solvable intervals. For more information, the reader is referred to [**1**, Chapters 1–9].

THEOREM 1.5. *Suppose* **A** *belongs to a locally finite variety which omits type* 1, *and* Con **A** *has a copy of* M_3 *as in Lemma* 1.3. *Then* θ *is abelian over* δ. *If* Con **A** *has a* 0, 1*-copy of* M_3, *then* **A** *is abelian.*

PROOF. It is an easy exercise to show that θ centralizes each α_i modulo δ, and hence each α_i is abelian over δ. It follows that each α_i is locally solvable over δ; hence [**1**, Lemma 7.4 and Corollary 7.5] $\theta = \bigvee_i \alpha_i$ is locally solvable over δ. Since **A** belongs to a locally finite variety which omits type 1, congruences in the interval $[\delta, \theta]$ permute [**1**, Theorem 7.12]). So Corollary 1.4 applies. □

2. In this section I construct, for every prime p, a finite nonabelian algebra whose congruence lattice has a 0,1-copy of M_{p+1}. In what follows, let p be a fixed prime.

DEFINITION 2.1. (1) $\mathscr{L}_p$ is the relational language containing the binary relation symbols $\theta_0, \theta_1, \ldots, \theta_p$.

(2) An M_{p+1}*-model* is a finite $\mathscr{L}_p$-structure $\mathbf{A} = \langle A; \theta_0^{\mathbf{A}}, \ldots, \theta_p^{\mathbf{A}} \rangle$ which satisfies:

(i) $|A| > 1$.

(ii) Each $\theta_i^{\mathbf{A}}$ is an equivalence relation on A.

(iii) $i \ne j$ implies $\theta_i^{\mathbf{A}} \vee \theta_j^{\mathbf{A}} = \nabla_A$ and $\theta_i^{\mathbf{A}} \cap \theta_j^{\mathbf{A}} = \Delta_A$.

Note that if **A** is an M_{p+1}-model and h is a homomorphism $h\colon \mathbf{A}^n \to \mathbf{A}$ for some $n \ge 1$, then $\langle A; h \rangle$ is a finite algebra whose congruence lattice has a 0, 1-copy of M_{p+1}.

DEFINITION 2.2. (1) The equivalence relations $\Theta_0, \ldots, \Theta_p$ on $\mathbf{Z}^2$ are defined by

$$\Theta_i = \{\langle (x,y),(x',y')\rangle : y' - y \equiv i(x'-x) \pmod p\} \quad \text{for } 0 \le i < p,$$
$$\Theta_p = \{\langle (x,y),(x',y')\rangle : x \equiv x' \pmod p\}.$$

(2) An M_{p+1}-model **A** is *standard* if $A \subseteq \mathbf{Z}^2$ and $\theta_i^{\mathbf{A}} \subseteq \Theta_i$ for each $i = 0, \ldots, p$; i.e. if the inclusion map $A \overset{\text{incl}}{\hookrightarrow} \mathbf{Z}^2$ is an injective homomorphism from **A** to $\langle \mathbf{Z}^2; \Theta_0, \ldots, \Theta_p \rangle$.

EXAMPLE 2.3. Define the $\mathscr{L}_p$-structure $\mathbf{V}$ by $V = \{0, 1, \ldots, p-1\}^2$ and $\theta_i^{\mathbf{V}} = \Theta_i \cap V^2$. Then $\mathbf{V}$ is a standard M_{p+1}-model. It corresponds to the congruence lattice of the 2-dimensional vector space over $GF(p)$.

Standard M_{p+1}-models can be visualized as labelled graphs in the plane. They can also be transformed and combined to make new standard M_{p+1} -models.

DEFINITION 2.4. Let $\mathbf{A}$, $\mathbf{B}$ be standard M_{p+1}-models, $v \in \mathbf{Z}^2$, and $\lambda \in \mathbf{Z} \setminus \{0\}$.

(1) The *translation of* $\mathbf{A}$ *by* v, denoted $\mathbf{A} + v$, is the $\mathscr{L}_p$-structure $\mathbf{A}'$ given by

$$A' = \{a + v\colon a \in A\},$$
$$\theta_i^{\mathbf{A}'} = \{\langle a+v, a'+v\rangle\colon \langle a, a'\rangle \in \theta_i^{\mathbf{A}}\}.$$

(2) The *dilation of* $\mathbf{A}$ *by* λ, denoted $\lambda\mathbf{A}$, is the $\mathscr{L}_p$-structure $\mathbf{A}'$ given by

$$A' = \{\lambda a\colon a \in A\},$$
$$\theta_i^{\mathbf{A}'} = \{\langle \lambda a, \lambda a'\rangle\colon \langle a, a'\rangle \in \theta_i^{\mathbf{A}}\}.$$

(3) Suppose $|A \cap B| = 1$. The *one point union* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cup \mathbf{B}$, is the $\mathscr{L}_p$-structure $\mathbf{A}'$ where $A' = A \cup B$ and each $\theta_i^{\mathbf{A}'}$ is the transitive closure of $\theta_i^{\mathbf{A}} \cup \theta_i^{\mathbf{B}}$.

(4) Suppose $u \in A$, $v \notin A$ and $\langle u, v\rangle \in \Theta_0 \cap \Theta_p$. The *perturbation of* $\mathbf{A}$ *sending* u *to* v is the $\mathscr{L}_p$-structure $\mathbf{A}'$ whose universe is $A' = (A \setminus \{u\}) \cup \{v\}$, and which is isomorphic to $\mathbf{A}$ via the isomorphism $f\colon \mathbf{A} \to \mathbf{A}'$ defined by

$$f(a) = \begin{cases} a & \text{if } a \neq u, \\ v & \text{if } a = u. \end{cases}$$

LEMMA 2.5. *All translations, dilations, one point unions and perturbations of standard M_{p+1}-models are standard M_{p+1}-models.* □

The next construction is rather more complicated. In what follows, π_i ($i = 1, 2$) are the projections of $\mathbf{Z}^2$ onto $\mathbf{Z}$.

DEFINITION 2.6. Let $\mathbf{A}$ be a standard M_{p+1}-model.

(1) The numbers $\underline{m}_1$, $\overline{m}_1$, $\underline{m}_2$, $\overline{m}_2 \in \mathbf{Z}$ are defined to be $\underline{m}_i = \min(\pi_i(A))$ and $\overline{m}_i = \max(\pi_i(A))$, $i = 1, 2$.

(2) The *lower-left* and *upper-right corners* of $\mathbf{A}$ are the points $\underline{m} = (\underline{m}_1, \underline{m}_2)$ and $\overline{m} = (\overline{m}_1, \overline{m}_2)$.

(3) The *diameter* of $\mathbf{A}$ is $\max(\overline{m}_1 - \underline{m}_1, \overline{m}_2 - \underline{m}_2)$.

(4) The *gauge* of $\mathbf{A}$ is

$$\min(\{|u - v|\colon u, v \in \pi_1(A), u \neq v\} \cup \{|u - v|\colon u, v \in \pi_2(A), u \neq v\}).$$

DEFINITION 2.7. Let $\mathbf{A}$, $\mathbf{B}$ be standard M_{p+1}-models. A *fiddle of* $\mathbf{B}$ *by* $\mathbf{A}$ consists of a family of standard M_{p+1}-models $\mathbf{B}_a$ and surjective homomorphisms $f_a\colon \mathbf{B} \to \mathbf{B}_a$, indexed by A, which satisfies the following:

(i) For each $i = 0, \ldots, p$ and each $\theta_i^{\mathbf{A}}$ equivalence class N there exists an equivalence relation θ_i^N on B such that for all $a \in N$ and b, $b' \in B$,

$$\langle f_a(b), f_a(b')\rangle \in \theta_i^{\mathbf{B}_a} \quad \text{iff} \quad \langle b, b'\rangle \in \theta_i^N.$$

(ii) For all $a \in A$ and $b \in B$, $\langle f_a(b), b\rangle \in \Theta_0 \cap \Theta_p$.

DEFINITION 2.8. Let **A**, **B** be standard M_{p+1}-models and suppose $\overline{f} = (\mathbf{B}_a, f_a)_{a\in A}$ is a fiddle of **B** by **A** which satisfies:

(i) The lower-left corner of each $\mathbf{B}_a$ is $\overline{0} = (0,0)$.

(ii) The gauge of **A** is greater than the diameter of each $\mathbf{B}_a$.

Then the *fiddled product* $\mathbf{A}\times_{\overline{f}}\mathbf{B}$ is the $\mathscr{L}_p$-structure **C** where $C = \bigcup\{B_a + a : a \in A\}$ and $\theta_i^{\mathbf{C}}$ is the transitive closure of

$$\bigcup\{\theta_i^{\mathbf{B}_a+a} : a \in A\} \cup \{\langle f_a(b)+a, f_{a'}(b)+a'\rangle : \langle a,a'\rangle \in \theta_i^{\mathbf{A}}, b \in B\}$$

LEMMA 2.9. *Every fiddled product of standard M_{p+1}-models is a standard M_{p+1}-model.*

PROOF. Let **A**, **B** and $\overline{f} = (\mathbf{B}_a, f_a)_{a\in A}$ be as in Definition 2.8, and let $\mathbf{C} = \mathbf{A}\times_{\overline{f}}\mathbf{B}$. Clearly $\theta_i^{\mathbf{C}} \vee \theta_j^{\mathbf{C}} = \nabla_C$ for $i \neq j$. For each $i = 0,\dots,p$ and each $\theta_i^{\mathbf{A}}$ equivalence class N, let θ_i^N be as in Definition 2.7. The conditions of Definition 2.8 imply that each $c \in C$ has a unique representation as $a+b$ for some $a \in A$ and $b \in B_a$. This fact and condition (i) of Definition 2.7 imply that for any a, $a' \in A$ and b, $b' \in B$, and any i,

$$\langle f_a(b)+a, f_{a'}(b')+a'\rangle \in \theta_i^{\mathbf{C}} \quad \text{iff} \quad \langle a,a'\rangle \in \theta_i^{\mathbf{A}} \text{ and } \langle b,b'\rangle \in \theta_i^N \text{ where } N = a/\theta_i^{\mathbf{A}}.$$

This implies that $\theta_i^{\mathbf{C}} \cap \theta_j^{\mathbf{C}} = \Delta_C$ for $i \neq j$, so **C** is an M_{p+1}-model. Condition (ii) of Definition 2.7 guarantees that **C** is standard. □

The constructions defined above also provide an abundance of homomorphisms.

LEMMA 2.10. *Let* **A**, **B**, **C** *be standard M_{p+1}-models.*

(1) *For each $v \in \mathbf{Z}^2$, the map $a \mapsto a+v$ is an isomorphism from* **A** *to* $\mathbf{A}+v$.

(2) *For each $\lambda \in \mathbf{Z}\setminus\{0\}$, the map $a \mapsto \lambda a$ is an isomorphism from* **A** *to* $\lambda\mathbf{A}$.

(3) *Suppose $|A\cap B| = 1$. Then the inclusion map $B \hookrightarrow A\cup B$ is an embedding of* **B** *into* $\mathbf{A}\cup\mathbf{B}$.

(4) *Suppose $A\cap B = A\cap C = \{a_0\}$ and $h\colon \mathbf{B}\to\mathbf{C}$ is a homomorphism such that $h(a_0) = a_0$. Then the map $\mathrm{id}_A \cup h$ is a homomorphism from $\mathbf{A}\cup\mathbf{B}$ to $\mathbf{A}\cup\mathbf{C}$.*

(5) *Suppose $\overline{f} = (\mathbf{B}_a, f_a)_{a\in A}$ is a fiddle of* **B** *by* **A** *which satisfies the conditions of Definition* 2.8. *Then:*

(i) *The map $(a,b) \mapsto f_a(b)+a$ is a homomorphism from $\mathbf{A}\times\mathbf{B}$ to $\mathbf{A}\times_{\overline{f}}\mathbf{B}$.*

(ii) *The map $a+b \mapsto a$, for $a \in A$ and $b \in B_a$, is a well-defined homomorphism from $\mathbf{A}\times_{\overline{f}}\mathbf{B}$ to* **A**.

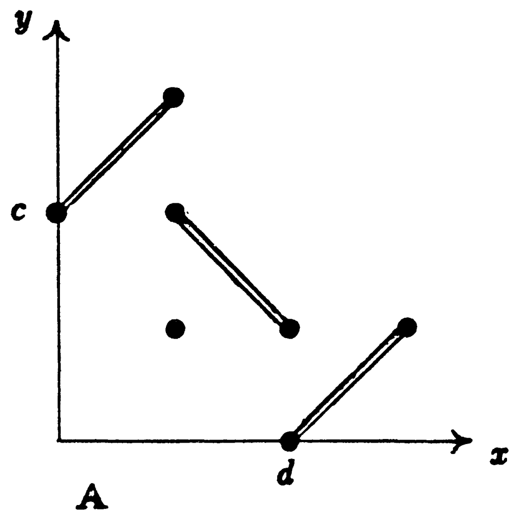
(6) *Let* **V** *be the standard M_{p+1}-model given in Example* 2.3. *For each $x \in \mathbf{Z}$ let $\mathrm{res}_p(x)$ denote the least residue of x mod p. Then the map $(x,y) \mapsto (\mathrm{res}_p(x), \mathrm{res}_p(y))$ is a homomorphism from $\langle \mathbf{Z}^2; \Theta_0,\dots,\Theta_p\rangle$ to* **V**.

PROOF. The only claim worth considering is 5(ii); it follows from the description of each $\theta_i^{\mathbf{C}}$ ($\mathbf{C} = \mathbf{A}\times_{\overline{f}}\mathbf{B}$) contained in the proof of Lemma 2.9. □

THEOREM 2.11. *Suppose there exists a standard M_{p+1}-model* **A** *and points c, $d \in A$ such that*

(i) *For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c,d\rangle \notin \theta_i^{\mathbf{A}} \circ \theta_j^{\mathbf{A}} \circ \theta_i^{\mathbf{A}}$.*

(ii) $\langle c,d\rangle \in \Theta_0 \cap \Theta_p$.



Then there exists a finite nonabelian algebra whose congruence lattice contains a $0,1$*-copy of* M_{p+1}.

EXAMPLE. If $p = 2$, then the $\mathscr{L}_2$-structure pictured below satisfies the hypotheses of the theorem. ($\theta_2^{\mathbf{A}}$ and $\theta_0^{\mathbf{A}}$ are the kernels of the first and second projections, respectively, while $\theta_1^{\mathbf{A}}$ is defined by the diagonal edges.)

PROOF OF THEOREM 2.11. It is easy to verify that the hypotheses imply $|A| \geq 4$. Pick elements a_0, $b_0 \in A$ such that a_0, b_0, c, d are all distinct. Via perturbations of a_0 and b_0, $\mathbf{A}$ can be fixed so that a_0 and b_0 are the lower-left and upper-right corners $\underline{m}$ and $\overline{m}$ of $\mathbf{A}$. (These corners will be used in the formation of one point unions.) It can also be assumed that $\underline{m} = \overline{0}$ (by translating $\mathbf{A}$ if necessary).

Let $\mathbf{V}$ be the standard M_{p+1}-model of Example 2.3; note that its lower-left corner is $\overline{0}$, and $\overline{0} \in V$. Thus $\mathbf{B} = \mathbf{A} \cup (\mathbf{V} + \overline{m})$ is defined, and $\mathbf{B}$ satisfies

(i)′ For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c, d\rangle \notin \theta_i^{\mathbf{B}} \circ \theta_j^{\mathbf{B}} \circ \theta_i^{\mathbf{B}}$.

Next define a fiddle of $\mathbf{B}$ by $\mathbf{V}$. For each $i = 0, \ldots, p$ let $\overline{\theta}_i$ be the symmetric transitive closure of $\theta_i^{\mathbf{B}} \cup \{\langle c, d\rangle\}$. For each $v \in V \setminus \{\overline{0}\}$ pick the unique $i \in \{0, \ldots, p\}$ such that $\langle v, \overline{0}\rangle \in \Theta_i$ and define $\mathbf{B}_v$ by

$$B_v = B,$$
$$\theta_j^{\mathbf{B}_v} = \begin{cases} \theta_j^{\mathbf{B}} & \text{if } j \neq i, \\ \overline{\theta}_i & \text{if } j = i. \end{cases}$$

Also let $f_v = \mathrm{id}_B$. Finally, define $\mathbf{B}_{\overline{0}}$ to be the $\mathscr{L}_p$-structure where $B_{\overline{0}} = B \setminus \{d\}$ and each $\theta_i^{\mathbf{B}_{\overline{0}}}$ is the symmetric transitive closure of

$$(\theta_i^{\mathbf{B}} \cap (B_{\overline{0}})^2) \cup \{\langle b, c\rangle : b \neq d, \langle b, d\rangle \in \theta_i^{\mathbf{B}}\}$$

Also define $f_{\overline{0}} \colon B \to B_{\overline{0}}$ by

$$f_{\overline{0}}(b) = \begin{cases} b & \text{if } b \neq d, \\ c & \text{if } b = d. \end{cases}$$

Property (i)′ implies that each $\mathbf{B}_v$ is an M_{p+1}-model while hypothesis (ii) implies it is standard. Clearly each f_v is a surjective homomorphism from $\mathbf{B}$ to $\mathbf{B}_v$, and

$\langle f_v(b), b\rangle \in \Theta_0 \cap \Theta_p$ for all $b \in B$. Suppose $i \in \{0, \dots, p\}$ and N is a $\theta_i^{\mathbf{V}}$ equivalence class. If $\overline{0} \notin N$, then the equivalence relation $\theta_i^N = \theta_i^{\mathbf{B}}$ satisfies condition (i) of Definition 2.7, while if $\overline{0} \in N$ then $\theta_i^N = \overline{\theta}_i$ satisfies the condition. Thus $\overline{f} = \langle \mathbf{B}_v, f_v\rangle_{v\in V}$ is a fiddle of $\mathbf{B}$ by $\mathbf{V}$.

$\overline{f}$ satisfies the first condition of Definition 2.8 but not the second. This is easily remedied. Let λ be an integer greater than the diameter of $\mathbf{B}$, and for each $v \in V$ let $\mathbf{B}'_{\lambda v} = \mathbf{B}_v$, $f'_{\lambda v} = f_v$ and $\overline{f}' = \langle \mathbf{B}'_{\lambda v}, f'_{\lambda v}\rangle_{\lambda v \in \lambda V}$. The gauge of $\lambda\mathbf{V}$ is λ, so $\lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B}$ is defined. Note that $\lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B}$ contains its lower-left corner, which is $\overline{0}$. Thus $\mathbf{C} = \mathbf{A} \cup [(\lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B}) + \overline{m}]$ is defined.

Now consider the following sequences of homomorphisms of $\mathscr{L}_p$-structures. (Unnamed maps are the canonical homomorphisms defined in Lemma 2.10.)

$$(1) \qquad \alpha\colon (\lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B}) + \overline{m} \cong \lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B} \to \lambda\mathbf{V} \cong \mathbf{V} \cong \mathbf{V} + \overline{m}.$$

Note that $\alpha(\overline{m}) = \overline{m}$; thus by Lemma 2.10(4), the map $\beta = \mathrm{id}_A \cup \alpha\colon \mathbf{C} \to \mathbf{B}$ is a homomorphism.

$$(2) \qquad \gamma\colon \mathbf{C} \overset{\text{incl}}{\hookrightarrow} \langle \mathbf{Z}^2; \Theta_0, \dots, \Theta_p\rangle \to \mathbf{V} \cong \lambda\mathbf{V},$$

$$(3) \qquad \delta\colon \lambda\mathbf{V} \times \mathbf{B} \to \lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B} \cong (\lambda\mathbf{V} \times_{\overline{f}'} \mathbf{B}) + \overline{m} \overset{\text{incl}}{\hookrightarrow} \mathbf{C},$$

$$(4) \qquad h\colon \mathbf{C} \times \mathbf{C} \overset{\gamma\times\beta}{\to} \lambda\mathbf{V} \times \mathbf{B} \overset{\delta}{\to} \mathbf{C}.$$

I claim that the finite algebra $\langle C; h\rangle$ is nonabelian. Indeed, γ is surjective (since $V + 2\overline{m} \subseteq B_{\overline{0}} + \overline{m} \subseteq C$), so there exist $a, b \in C$ such that $\gamma(a) = \overline{0}$ while $\gamma(b) = \lambda v \neq \overline{0}$. It is easy to check that

$$h(a, c) = f_{\overline{0}}(c) + \overline{m} = f_{\overline{0}}(d) + \overline{m} = h(a, d)$$

but

$$h(b, c) = f_v(c) + \lambda v + \overline{m} \neq f_v(d) + \lambda v + \overline{m} = h(b, d). \quad \square$$

LEMMA 2.12. *There exists a standard M_{p+1}-model $\mathbf{A}$ and points $c, d \in A$ satisfying the hypotheses of Theorem* 2.11.

PROOF. It suffices to assume $p \geq 3$. Let $\mathbf{D}$ be the $\mathscr{L}_p$-structure given by

$$D = V \setminus \{\overline{0}\},$$
$$\theta_i^{\mathbf{D}} = (\theta_i^{\mathbf{V}} \setminus N^2) \cup \Delta_D, \quad \text{where } N = \overline{0}/\theta_i^{\mathbf{V}}.$$

It is not difficult to show that $\theta_i^{\mathbf{D}} \circ \theta_j^{\mathbf{D}} \circ \theta_i^{\mathbf{D}} \circ \theta_j^{\mathbf{D}} = \nabla_D$ if $i \neq j$, so $\mathbf{D}$ is a standard M_{p+1}-model. For each $i = 0, \dots, p-1$ let u_i, v_i be the unique elements of $\{0, \dots, p-1\}$ such that $\langle (1, u_i), \overline{0}\rangle, \langle (2, v_i), \overline{0}\rangle \in \theta_i^{\mathbf{V}}$. Let $c = (0, p)$ and $d = (p, 0)$. Now let $\mathbf{A}$ be the $\mathscr{L}_p$-structure given by

$$A = D \cup \{c, d\},$$
$$\theta_i^{\mathbf{A}} = \text{the reflexive symmetric closure of}$$
$$\theta_i^{\mathbf{D}} \cup \{\langle c, (1, u_i)\rangle, \langle d, (2, v_i)\rangle\}, \quad \text{for } i = 0, \dots, p-1,$$
$$\theta_p^{\mathbf{A}} = \theta_p^{\mathbf{D}} \cup \{\langle c, c\rangle, \langle d, d\rangle\}.$$

Then $\mathbf{A}$ is a standard M_{p+1}-model which satisfies the hypotheses of Theorem 2.11. $\square$

COROLLARY 2.13. *For every $n \geq 1$ there is a finite nonabelian algebra whose congruence lattice contains a* $0,1$*-copy of* M_n. □

Corollary 2.13 prompts the following question: Does there exist *any* finite lattice L which forces finite algebras $\mathbf{A}$ to be abelian whenever $\operatorname{Con}\mathbf{A}$ has a $0,1$-copy of L? By the embedding theorem of P. Pudlák and J. Tůma [**3**], there exists a lattice which satisfies this condition if and only if some finite partition lattice Π_n satisfies the condition.

REFERENCES

1. R. McKenzie and D. Hobby, *The structure of finite algebras* (*tame congruence theory*), Contemp. Math., Amer. Math. Soc., Providence, R.I. (forthcoming).
2. R. McKenzie, G. McNulty and W. Taylor, *Algebras, lattices, varieties*, vol. I, Wadsworth & Brooks/Cole, 1987.
3. P. Pudlák and J. Tůma, *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis **10** (1980), 74–95.

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1