$M_n$ as a 0, 1-sublattice of $\text{Con} A$ does not force the term condition

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Abstract. For every $n \geq 3$ there exists a finite nonabelian algebra whose congruence lattice has $M_n$ as a 0, 1-sublattice. This answers a question of R. McKenzie and D. Hobby.

Definition 0.1. Suppose $L$ and $L_1$ are bounded lattices. A copy of $L$ (in $L_1$) is any sublattice $L'$ of $L_1$ which is isomorphic to $L$. $L'$ is a 0, 1-copy of $L$ if it includes the least and greatest elements of $L_1$; in this case $L'$ is also called a 0, 1-sublattice of $L_1$.

Definition 0.2. For $n \geq 1$, $M_n$ is the finite lattice of height 2 having exactly $n$ atoms. For example, $M_6$ is

Suppose $G$ is a group and $N(G)$ is its lattice of normal subgroups. There is a trivial proof, using the commutator operation on normal subgroups, that if $N(G)$ has a 0,1-copy of $M_3$ then $G$ is abelian. This same proof extends, via the general commutator theory of universal algebra, to any algebra $A$ belonging to a variety whose congruence lattices satisfy the modular law. Here $N(G)$ is replaced by $\text{Con} A$ (the lattice of congruence relations of $A$), while ‘abelian’ means the following ‘term condition’:

Definition 0.3. An algebra $A$ is abelian if for every $n \geq 1$, every $(n + 1)$-ary term $t(x, y_1, \ldots, y_n)$ in the language of $A$, and all $a, b, c_1, \ldots, c_n, d_1, \ldots, d_n \in A$,

$$t^A(a, c) = t^A(a, d) \quad \text{iff} \quad t^A(b, c) = t^A(b, d).$$

In their forthcoming book on tame congruence theory [1], R. McKenzie and D. Hobby ask whether the above phenomenon—$\text{Con} A$ having $M_3$ as a 0, 1-sublattice

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forcing $\mathbf{A}$ to be abelian—holds for all finite algebras (without assuming congruence modularity); and if not, whether there is any $n > 3$ for which having $M_n$ as a 0, 1-sublattice suffices. The purpose of this paper is to note that the answer to the first question is positive (without requiring finiteness) in case $\mathbf{A}$ belongs to a locally finite variety which 'omits type 1', while the answers to both questions are negative in general.

1. In this section I prove the positive answer for algebras belonging to locally finite varieties which omit type 1. Definition 1.1 is taken from [2], and Lemma 1.4 is a slight improvement of Lemma 4.153 found there.

**Definition 1.1.** Suppose $\alpha$, $\beta$, $\delta$ are congruences of an algebra $\mathbf{A}$.

(1) $\alpha$ centralizes $\beta$ modulo $\delta$ if, for every $n \geq 1$, every term $t(x, y_1, \ldots, y_n)$ in the language of $\mathbf{A}$, and all $\langle a, b \rangle \in \alpha$, $\langle c_1, d_1 \rangle, \ldots, \langle c_n, d_n \rangle \in \beta$,

$$t^\mathbf{A}(a, \overrightarrow{c}) \equiv t^\mathbf{A}(a, \overrightarrow{d}) \quad \text{iff} \quad t^\mathbf{A}(b, \overrightarrow{c}) \equiv t^\mathbf{A}(b, \overrightarrow{d}).$$

(2) If $\alpha \geq \delta$, then $\alpha$ is abelian over $\delta$ if $\alpha$ centralizes itself modulo $\delta$. (Thus $\mathbf{A}$ is abelian iff $\nabla_A$ is abelian over $\Delta_A$.)

**Definition 1.2.** Suppose $\delta$, $\theta$ are congruences of $\mathbf{A}$ with $\theta \geq \delta$.

(1) $\theta$ is solvable over $\delta$ if there exist $\alpha_0, \ldots, \alpha_k \in \Con \mathbf{A}$ such that $\alpha_0 = \delta$, $\alpha_k = \theta$, and $\alpha_i+1$ is abelian over $\alpha_i$ for each $i < k$.

(2) $\theta$ is locally solvable over $\delta$ if in every finitely generated subalgebra $\mathbf{B} \leq \mathbf{A}$, $\theta \cap B^2$ is solvable over $\delta \cap B^2$.

**Lemma 1.3.** Suppose $\mathbf{A}$ is an algebra and $\Con \mathbf{A}$ has a copy of $M_3$, as in the picture below.

![Diagram](attachment:image.png)

If $\alpha_2 \circ \alpha_3 \circ \alpha_2 = \theta$, then $\alpha_1$ centralizes $\theta$ modulo $\alpha_2$, and hence modulo $\delta$.

**Proof.** Let $t(x, y_1, \ldots, y_n)$ be a term in the language of $\mathbf{A}$, and $\langle a, b \rangle \in \alpha_1$, $\langle c_1, d_1 \rangle, \ldots, \langle c_n, d_n \rangle \in \theta$. Pick $c'_i$, $d'_i$ such that $c_i \equiv c'_i \equiv d'_i \equiv d_i$ for each $i = 1, \ldots, n$. Now suppose $t^\mathbf{A}(a, \overrightarrow{c}) \equiv t^\mathbf{A}(a, \overrightarrow{d})$. Then

$$t^\mathbf{A}(a, \overrightarrow{c'}) \equiv t^\mathbf{A}(a, \overrightarrow{c}) \equiv t^\mathbf{A}(a, \overrightarrow{d}) \equiv t^\mathbf{A}(a, \overrightarrow{d'}) \quad \text{and} \quad t^\mathbf{A}(a, \overrightarrow{c'}) \equiv t^\mathbf{A}(a, \overrightarrow{d'})$$

so

$$(t^\mathbf{A}(a, \overrightarrow{c'}), t^\mathbf{A}(a, \overrightarrow{d'})) \in \alpha_2 \cap \alpha_3 \subseteq \alpha_1.$$ 

Similarly, $\langle t^\mathbf{A}(b, \overrightarrow{c'}), t^\mathbf{A}(b, \overrightarrow{d'}) \rangle \in \alpha_1 \cap \alpha_3 \subseteq \alpha_2$. So $t^\mathbf{A}(b, \overrightarrow{c}) \equiv t^\mathbf{A}(b, \overrightarrow{d})$, which proves that $\alpha_1$ centralizes $\theta$ modulo $\alpha_2$.

Now $\alpha_1$ automatically centralizes $\theta$ modulo $\alpha_1 \cap \theta = \alpha_1$. It follows that $\alpha_1$ centralizes $\theta$ modulo $\alpha_2 \cap \alpha_1 = \delta$. □
COROLLARY 1.4. Suppose Con A has a copy of M₃ as in the previous lemma. If for some \( \{i_1, i_2\} = \{2, 3\} \) and \( \{j_1, j_2\} = \{1, 3\} \) the condition
\[
\alpha_{i_1} \circ \alpha_{i_2} \circ \alpha_{i_1} = \alpha_{j_1} \circ \alpha_{j_2} \circ \alpha_{j_1} = \theta
\]
is met, then \( \theta \) is abelian over \( \delta \).

PROOF. By the previous lemma, \( \alpha_1 \) and \( \alpha_2 \) both centralize \( \theta \) modulo \( \delta \). It follows that \( \alpha_1 \vee \alpha_2 = \theta \) centralizes \( \theta \) modulo \( \delta \). \( \square \)

The concept of a locally finite variety \( V \) omitting type 1 is central to tame congruence theory. One way to define this concept is as follows: \( V \) omits type 1 if for every \( A \in V \) and every \( \theta \in \text{Con} A \) different from \( \Delta_A \), there is an \( (n+1) \)-ary term \( t(x, \overrightarrow{y}) \) in the language of \( V \), and there exist elements \( a \equiv b, c_i \equiv d_i \equiv e_i \) (1 \( \leq i \leq n \)) of \( A \), such that \( t^A(a, \overrightarrow{c}) = t^A(b, \overrightarrow{d}) \) but \( t^A(a, \overrightarrow{e}) \neq t^A(b, \overrightarrow{e}) \). Most interesting locally finite varieties, including those whose congruence lattices satisfy some nontrivial lattice identity, omit type 1. Omitting type 1 is equivalent to the variety satisfying some nontrivial idempotent Malcev condition, and also to the permuting of congruences in locally solvable intervals. For more information, the reader is referred to [1, Chapters 1–9].

THEOREM 1.5. Suppose \( A \) belongs to a locally finite variety which omits type 1, and \( \text{Con} A \) has a copy of \( M_3 \) as in Lemma 1.3. Then \( \theta \) is abelian over \( \delta \). If \( \text{Con} A \) has a 0,1-copy of \( M_{p+1} \), then \( A \) is abelian.

PROOF. It is an easy exercise to show that \( \theta \) centralizes each \( \alpha_i \) modulo \( \delta \), and hence each \( \alpha_i \) is abelian over \( \delta \). It follows that each \( \alpha_i \) is locally solvable over \( \delta \); hence [1, Lemma 7.4 and Corollary 7.5] \( \theta = \bigvee_i \alpha_i \) is locally solvable over \( \delta \). Since \( A \) belongs to a locally finite variety which omits type 1, congruences in the interval \( [\delta, \theta] \) permute [1, Theorem 7.12]). So Corollary 1.4 applies. \( \square \)

2. In this section I construct, for every prime \( p \), a finite nonabelian algebra whose congruence lattice has a 0,1-copy of \( M_{p+1} \). In what follows, let \( p \) be a fixed prime.

DEFINITION 2.1. (1) \( \mathcal{L}_p \) is the relational language containing the binary relation symbols \( \theta_0, \theta_1, \ldots, \theta_p \).

(2) An \( M_{p+1} \)-model is a finite \( \mathcal{L}_p \)-structure \( A = \langle A; \theta_0^A, \ldots, \theta_p^A \rangle \) which satisfies:

(i) \( |A| > 1 \).

(ii) Each \( \theta_i^A \) is an equivalence relation on \( A \).

(iii) \( i \neq j \) implies \( \theta_i^A \vee \theta_j^A = \nabla_A \) and \( \theta_i^A \cap \theta_j^A = \Delta_A \).

Note that if \( A \) is an \( M_{p+1} \)-model and \( h \) is a homomorphism \( h \colon A^n \to A \) for some \( n \geq 1 \), then \( \langle A; h \rangle \) is a finite algebra whose congruence lattice has a 0,1-copy of \( M_{p+1} \).

DEFINITION 2.2. (1) The equivalence relations \( \Theta_0, \ldots, \Theta_p \) on \( \mathbb{Z}^2 \) are defined by
\[
\Theta_i = \{(x, y), (x', y') \} : y' - y \equiv i(x' - x) \pmod{p}\}
\]
for \( 0 \leq i < p \),
\[
\Theta_p = \{(x, y), (x', y') \} : x \equiv x' \pmod{p}\}.
\]

(2) An \( M_{p+1} \)-model \( A \) is standard if \( A \subseteq \mathbb{Z}^2 \) and \( \theta_i^A \subseteq \Theta_i \) for each \( i = 0, \ldots, p \); i.e. if the inclusion map \( A^\text{incl} \hookrightarrow \mathbb{Z}^2 \) is an injective homomorphism from \( A \) to \( (\mathbb{Z}^2; \Theta_0, \ldots, \Theta_p) \).
EXAMPLE 2.3. Define the \( L_i \)-structure \( V \) by \( V = \{0, 1, \ldots, p - 1\}^2 \) and \( \theta_i^V = \Theta_i \cap V^2 \). Then \( V \) is a standard \( M_{p+1} \)-model. It corresponds to the congruence lattice of the 2-dimensional vector space over \( GF(p) \).

Standard \( M_{p+1} \)-models can be visualized as labelled graphs in the plane. They can also be transformed and combined to make new standard \( M_{p+1} \)-models.

DEFINITION 2.4. Let \( A, B \) be standard \( M_{p+1} \)-models, \( v \in Z^2 \), and \( \lambda \in Z \setminus \{0\} \).

(1) The translation of \( A \) by \( v \), denoted \( A + v \), is the \( L_i \)-structure \( A' \) given by

\[ A' = \{a + v : a \in A\}, \]
\[ \theta_i^{A'} = \{(a + v, a' + v) : (a, a') \in \theta_i^A\}. \]

(2) The dilation of \( A \) by \( \lambda \), denoted \( \lambda A \), is the \( L_i \)-structure \( A' \) given by

\[ A' = \{\lambda a : a \in A\}, \]
\[ \theta_i^{A'} = \{(\lambda a, \lambda a') : (a, a') \in \theta_i^A\}. \]

(3) Suppose \( |A \cap B| = 1 \). The one point union of \( A \) and \( B \), denoted by \( A \cup B \), is the \( L_i \)-structure \( A' \) where \( A' = A \cup B \) and each \( \theta_i^{A'} \) is the transitive closure of \( \theta_i^A \cup \theta_i^B \).

(4) Suppose \( u \in A, v \notin A \) and \( (u, v) \in \Theta_0 \cap \Theta_p \). The perturbation of \( A \) sending \( u \) to \( v \) is the \( L_i \)-structure \( A' \) whose universe is \( A' = (A \setminus \{u\}) \cup \{v\} \), and which is isomorphic to \( A \) via the isomorphism \( f : A \to A' \) defined by

\[ f(a) = \begin{cases} a & \text{if } a \neq u, \\ v & \text{if } a = u. \end{cases} \]

LEMMA 2.5. All translations, dilations, one point unions and perturbations of standard \( M_{p+1} \)-models are standard \( M_{p+1} \)-models. \( \square \)

The next construction is rather more complicated. In what follows, \( \pi_i (i = 1, 2) \) are the projections of \( Z^2 \) onto \( Z \).

DEFINITION 2.6. Let \( A \) be a standard \( M_{p+1} \)-model.

(1) The numbers \( m_1, \overline{m}_1, m_2, \overline{m}_2 \in Z \) are defined to be \( m_i = \min(\pi_i(A)) \) and \( \overline{m}_i = \max(\pi_i(A)) \), \( i = 1, 2 \).

(2) The lower-left and upper-right corners of \( A \) are the points \( m = (m_1, m_2) \) and \( \overline{m} = (\overline{m}_1, \overline{m}_2) \).

(3) The diameter of \( A \) is \( \max(\overline{m}_1 - m_1, \overline{m}_2 - m_2) \).

(4) The gauge of \( A \) is
\[ \min(\{|u - v| : u, v \in \pi_1(A), u \neq v\} \cup \{|u - v| : u, v \in \pi_2(A), u \neq v\}). \]

DEFINITION 2.7. Let \( A, B \) be standard \( M_{p+1} \)-models. A fiddle of \( B \) by \( A \) consists of a family of standard \( M_{p+1} \)-models \( B_a \) and surjective homomorphisms \( f_a : B \to B_a \), indexed by \( A \), which satisfies the following:

(i) For each \( i = 0, \ldots, p \) and each \( \theta_i^B \) equivalence class \( N \) there exists an equivalence relation \( \theta_i^N \) on \( B \) such that for all \( a \in N \) and \( b, b' \in B \),
\[ \langle f_a(b), f_a(b') \rangle \in \theta_i^{B_a} \iff \langle b, b' \rangle \in \theta_i^N. \]

(ii) For all \( a \in A \) and \( b \in B \), \( \langle f_a(b), b \rangle \in \Theta_0 \cap \Theta_p \).
DEFINITION 2.8. Let $A$, $B$ be standard $M_{p+1}$-models and suppose $\tilde{f} = (B_a, f_a)_{a \in A}$ is a fiddle of $B$ by $A$ which satisfies:

(i) The lower-left corner of each $B_a$ is $0 = (0,0)$.

(ii) The gauge of $A$ is greater than the diameter of each $B_a$.

Then the fiddled product $A \times_{\tilde{f}} B$ is the $\mathcal{L}_p$-structure $C$ where $C = \bigcup \{B_a + a : a \in A\}$ and $\theta^C$ is the transitive closure of

$$\bigcup \{\theta^{B_a + a}_i : a \in A\} \cup \{(f_a(b) + a, f'_a(b) + a') : (a, a') \in \theta^A_i, b \in B\}$$

LEMMA 2.9. Every fiddled product of standard $M_{p+1}$-models is a standard $M_{p+1}$-model.

PROOF. Let $A$, $B$ and $\tilde{f} = (B_a, f_a)_{a \in A}$ be as in Definition 2.8, and let $C = A \times_{\tilde{f}} B$. Clearly $\theta^C \cap \theta^C = \nabla_C$ for $i \neq j$. For each $i = 0, \ldots, p$ and each $\theta^A_i$ equivalence class $N$, let $\theta^N_i$ be as in Definition 2.7. The conditions of Definition 2.8 imply that each $c \in C$ has a unique representation as $a + b$ for some $a \in A$ and $b \in B_a$. This fact and condition (i) of Definition 2.7 imply that for any $a, a' \in A$ and $b, b' \in B$, and any $i$,

$$\langle f_a(b) + a, f'_a(b') + a' \rangle \in \theta^C_i \iff \langle a, a' \rangle \in \theta^A_i \text{ and } \langle b, b' \rangle \in \theta^N_i \text{ where } N = a/\theta^A_i.$$

This implies that $\theta^C_i \cap \theta^C_j = \Delta_C$ for $i \neq j$, so $C$ is an $M_{p+1}$-model. Condition (ii) of Definition 2.7 guarantees that $C$ is standard. \qed

The constructions defined above also provide an abundance of homomorphisms.

LEMMA 2.10. Let $A$, $B$, $C$ be standard $M_{p+1}$-models.

(1) For each $v \in \mathbb{Z}^2$, the map $a \mapsto a + v$ is an isomorphism from $A$ to $A + v$.

(2) For each $\lambda \in \mathbb{Z} \setminus \{0\}$, the map $a \mapsto \lambda a$ is an isomorphism from $A$ to $\lambda A$.

(3) Suppose $|A \cap B| = 1$. Then the inclusion map $B \hookrightarrow A \cup B$ is an embedding of $B$ into $A \cup B$.

(4) Suppose $A \cap B = A \cap C = \{a_0\}$ and $h : B \to C$ is a homomorphism such that $h(a_0) = a_0$. Then the map $id_A \cup h$ is a homomorphism from $A \cup B$ to $A \cup C$.

(5) Suppose $\tilde{f} = (B_a, f_a)_{a \in A}$ is a fiddle of $B$ by $A$ which satisfies the conditions of Definition 2.8. Then:

(i) The map $(a, b) \mapsto f_a(b) + a$ is a homomorphism from $A \times B$ to $A \times_{\tilde{f}} B$.

(ii) The map $a + b \mapsto a$, for $a \in A$ and $b \in B_a$, is a well-defined homomorphism from $A \times_{\tilde{f}} B$ to $A$.

(6) Let $V$ be the standard $M_{p+1}$-model given in Example 2.3. For each $x \in \mathbb{Z}$ let $\text{res}_p(x)$ denote the least residue of $x$ mod $p$. Then the map $(x, y) \mapsto (\text{res}_p(x), \text{res}_p(y))$ is a homomorphism from $(\mathbb{Z}^2; \Theta_0, \ldots, \Theta_p)$ to $V$.

PROOF. The only claim worth considering is 5(ii); it follows from the description of each $\theta^C_i (C = A \times_{\tilde{f}} B)$ contained in the proof of Lemma 2.9. \qed

THEOREM 2.11. Suppose there exists a standard $M_{p+1}$-model $A$ and points $c, d \in A$ such that

(i) For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c, d \rangle \notin \theta^A_i \circ \theta^A_j$.

(ii) $\langle c, d \rangle \in \Theta_0 \cap \Theta_p$.

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Then there exists a finite nonabelian algebra whose congruence lattice contains a 
0,1-copy of $M_{p+1}$.

**EXAMPLE.** If $p = 2$, then the $L_2$-structure pictured below satisfies the hypo-
theses of the theorem. ($\theta^A_2$ and $\theta^A_0$ are the kernels of the first and second projections, 
respectively, while $\theta^A_1$ is defined by the diagonal edges.)

**PROOF OF THEOREM 2.11.** It is easy to verify that the hypotheses imply 
$|A| \geq 4$. Pick elements $a_0, b_0 \in A$ such that $a_0, b_0, c, d$ are all distinct. Via 
perturbations of $a_0$ and $b_0$, $A$ can be fixed so that $a_0$ and $b_0$ are the lower-left and 
upper-right corners $m$ and $\overline{m}$ of $A$. (These corners will be used in the formation 
of one point unions.) It can also be assumed that $\overline{m} = \overline{0}$ (by translating $A$ if 
necessary).

Let $V$ be the standard $M_{p+1}$-model of Example 2.3; note that its lower-left 
corner is $\overline{0}$, and $\overline{0} \in V$. Thus $B = A \cup (V + \overline{m})$ is defined, and $B$ satisfies

(i)' For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c, d \rangle \notin \theta^B_i \circ \theta^B_j \circ \theta^B_i$.

Next define a fiddle of $B$ by $V$. For each $i = 0, \ldots, p$ let $\bar{\theta}_i$ be the symmetric 
transitive closure of $\theta^B_i \cup \{\langle c, d \rangle\}$. For each $v \in V \setminus \{\overline{0}\}$ pick the unique $i \in \{0, \ldots, p\}$ 
such that $\langle v, \overline{0} \rangle \in \Theta_i$ and define $B_v$ by

$$B_v = B,$$

$$\theta^B_j = \begin{cases} \theta^B_j & \text{if } j \neq i, \\ \bar{\theta}_i & \text{if } j = i. \end{cases}$$

Also let $f_v = \text{id}_B$. Finally, define $B_0$ to be the $L_p$-structure where $B_0 = B \setminus \{d\}$ 
and each $\theta^B_i$ is the symmetric transitive closure of

$$\left(\theta^B_i \cap (B_0)^2\right) \cup \{\langle b, c \rangle : b \neq d, \langle b, d \rangle \in \theta^B_i\}$$

Also define $f_0: B \to B_0$ by

$$f_0(b) = \begin{cases} b & \text{if } b \neq d, \\ c & \text{if } b = d. \end{cases}$$

Property (i)' implies that each $B_v$ is an $M_{p+1}$-model while hypothesis (ii) implies 
it is standard. Clearly each $f_v$ is a surjective homomorphism from $B$ to $B_v$, and
\[ (f_v(b), b) \in \Theta_0 \cap \Theta_p \text{ for all } b \in B. \] Suppose \( i \in \{0, \ldots, p\} \) and \( N \) is a \( \theta^V_i \) equivalence class. If \( \bar{0} \notin N \), then the equivalence relation \( \theta^N_i = \theta^B_i \) satisfies condition (i) of Definition 2.7, while if \( \bar{0} \in N \) then \( \theta^N_i = \bar{0} \) satisfies the condition. Thus \( \bar{f} = \langle B, f_v \rangle_{v \in V} \) is a fiddle of \( B \) by \( V \).

\( \bar{f} \) satisfies the first condition of Definition 2.8 but not the second. This is easily remedied. Let \( \lambda \) be an integer greater than the diameter of \( B \), and for each \( v \in V \) let \( B_{AV} = B, f'_{AV} = f_v \) and \( \bar{f} = \langle B', f'_{AV} \rangle_{\lambda \in \lambda V} \). The gauge of \( \lambda V \) is \( \lambda \), so \( \lambda V \times \gamma' B \) is defined. Note that \( \lambda V \times \gamma' B \) contains its lower-left corner, which is \( \bar{0} \).

Thus \( C = A \cup [(\lambda V \times \gamma' B) + \bar{m}] \) is defined.

Now consider the following sequences of homomorphisms of \( L_p \)-structures. (Unamed maps are the canonical homomorphisms defined in Lemma 2.10.)

\[
(1) \quad \alpha : (\lambda V \times \gamma' B) + \bar{m} \equiv \lambda V \times \gamma' B \rightarrow \lambda V \equiv V \equiv V + \bar{m}.
\]

Note that \( \alpha(\bar{m}) = \bar{m} \); thus by Lemma 2.10(4), the map \( \beta = \text{id}_A \cup \alpha : C \rightarrow B \) is a homomorphism.

\[
(2) \quad \gamma : C \xleftarrow{\text{incl}} \langle \mathbb{Z}^2; \Theta_0, \ldots, \Theta_p \rangle \rightarrow V \equiv \lambda V,
\]

\[
(3) \quad \delta : \lambda V \times B \rightarrow \lambda V \times \gamma' B \equiv (\lambda V \times \gamma' B) + \bar{m} \xrightarrow{\text{incl}} C,
\]

\[
(4) \quad h : C \times C \xrightarrow{\gamma \times \delta} \lambda V \times B \xrightarrow{\delta} C.
\]

I claim that the finite algebra \( \langle C; h \rangle \) is nonabelian. Indeed, \( \gamma \) is surjective (since \( V + 2\bar{m} \subseteq B_0 + \bar{m} \subseteq C \)), so there exist \( a, b \in C \) such that \( \gamma(a) = \bar{0} \) while \( \gamma(b) = \lambda v \neq \bar{0} \). It is easy to check that

\[
h(a, c) = f_0(c) + \bar{m} = f_0(d) + \bar{m} = h(a, d)
\]

but

\[
h(b, c) = f_v(c) + \lambda v + \bar{m} \neq f_v(d) + \lambda v + \bar{m} = h(b, d). \quad \Box
\]

**LEMMA 2.12.** There exists a standard \( M_{p+1} \)-model \( A \) and points \( c, d \in A \) satisfying the hypotheses of Theorem 2.11.

**PROOF.** It suffices to assume \( p \geq 3 \).

Let \( D \) be the \( L_p \)-structure given by

\[
D = V \setminus \{\bar{0}\},
\]

\[
\theta^D_i = (\theta^V_i \setminus N^2) \cup \Delta_D, \quad \text{where } N = \bar{0}/\theta^V_i.
\]

It is not difficult to show that \( \theta^D_i \circ \theta^D_j \circ \theta^D_i \circ \theta^D_j = \nabla_D \) if \( i \neq j \), so \( D \) is a standard \( M_{p+1} \)-model. For each \( i = 0, \ldots, p-1 \) let \( u_i, v_i \) be the unique elements of \( \{0, \ldots, p-1\} \) such that \( \langle (1, u_i), \bar{0} \rangle, \langle (2, v_i), \bar{0} \rangle \in \theta^V_i \). Let \( c = (0, p) \) and \( d = (p, 0) \).

Now let \( A \) be the \( L_p \)-structure given by

\[
A = D \cup \{c, d\},
\]

\[
\theta^A_i = \text{the reflexive symmetric closure of}
\]

\[
\theta^D_i \cup \{\langle c, (1, u_i) \rangle, \langle d, (2, v_i) \rangle\}, \quad \text{for } i = 0, \ldots, p-1,
\]

\[
\theta^A_p = \theta^D_p \cup \{\langle c, c \rangle, \langle d, d \rangle\}.
\]

Then \( A \) is a standard \( M_{p+1} \)-model which satisfies the hypotheses of Theorem 2.11. \( \Box \)
COROLLARY 2.13. For every \( n \geq 1 \) there is a finite nonabelian algebra whose congruence lattice contains a 0,1-copy of \( M_n \). \( \square \)

Corollary 2.13 prompts the following question: Does there exist any finite lattice \( L \) which forces finite algebras \( A \) to be abelian whenever \( \text{Con} A \) has a 0,1-copy of \( L \)? By the embedding theorem of P. Pudlák and J. Tůma [3], there exists a lattice which satisfies this condition if and only if some finite partition lattice \( \Pi_n \) satisfies the condition.

REFERENCES


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