UNCOUNTABLE SIS IN RS VARIETIES IN COUNTABLE SIGNATURE APRIL 6, 2018

Holy Grail: V variety, σ signature.

$$\left.\begin{array}{c} V \text{ locally finite, } \sigma \text{ finite,} \\ \text{arbitrarily large finite SIs} \end{array}\right\} \overset{?}{\Longrightarrow} \text{infinite SI.}$$

Our goal: to prove

Theorem (McKenzie, Shelah 1974). σ countable, uncountable SI $\Rightarrow$ SI of size $\geq 2^\omega$.

Given V, let $V_{si} = \{\text{all SIs in } V\}$, $\operatorname{Spec}(V) = \{|A| : \mathbf{A} \in V_{si}\}$.

Fact 1. Can assume V is residually small; hence every SI has size $\leq 2^\omega$.

Definition. V is *prepared* if σ has constants $0, 1$ such that $\forall \mathbf{A} \in V_{si}\ \exists \mathbf{B} \in V_{si}$ with $|B| = |A|$ and $(0^{\mathbf{B}}, 1^{\mathbf{B}}) \in \mu \setminus 0_B$.

Fact 2. Can assume V is prepared.

Proof. Let Σ be a set of (universally quantified) identities axiomatizing V.
Let $\sigma^+ = \sigma \cup \{0, 1\}$.
Let $V^+ =$ the variety in signature σ^+ axiomatized by Σ.
V^+ is prepared, and $\operatorname{Spec}(V^+) = \operatorname{Spec}(V)$.
So if the theorem is true for V^+, then it's also true for V. □

From now on, assume V is residually small and prepared.

Definition. $V_{si}^{01} = \{\mathbf{A} \in V_{si} : (0^{\mathbf{A}}, 1^{\mathbf{A}}) \in \mu \setminus 0_A\}$.

Definition. A *primitive positive* (pp) formula is a first-order formula $\varphi(\mathbf{x})$ of the form $\exists \mathbf{y} \bigwedge_{i=1}^{n} \alpha_i(\mathbf{x}, \mathbf{y})$ where each α_i is atomic (i.e., an equation between terms).

Example: a *principal congruence formula* $\pi(x_1, x_2, x_3, x_4)$, which encodes a particular way that (x_3, x_4) might be forced to belong to $\operatorname{Cg}(x_1, x_2)$.

Let $\Pi(\sigma) =$ the set of all principal congruence formulas in signature σ.

V_{si}^{01} is axiomatized by

$$\Sigma \ \cup\ \{0 \neq 1\} \ \cup\ \forall x, y \left[x \neq y \to \bigvee_{\pi \in \Pi(\sigma)} \pi(x, y, 0, 1) \right].$$

Proposition 1. *V_{si}^{01} is closed under unions of chains.*

Proof. It is enough to consider chains $\{\mathbf{A}_\alpha : \alpha < \lambda\} \subseteq V_{si}^{01}$ indexed by an ordinal λ, where $\alpha < \beta$ implies $\mathbf{A}_\alpha \leq \mathbf{A}_\beta$. Let $\mathbf{A} = \bigcup_\alpha \mathbf{A}_\alpha$. Clearly $\mathbf{A} \in V$ and $\mathbf{A} \models 0 \neq 1$. Suppose $a, b \in A$ with $a \neq b$. Pick α with $a, b \in A_\alpha$. $\mathbf{A}_\alpha \in V_{si}^{01}$ and $a \neq b$, so $(0,1) \in \mathrm{Cg}^{\mathbf{A}_\alpha}(a,b)$. Hence there exists $\pi \in \Pi(\sigma)$ such that $\mathbf{A}_\alpha \models \pi(a,b,0,1)$. As $\mathbf{A}_\alpha \leq \mathbf{A}$ and π is existential, we get $\mathbf{A} \models \pi(a,b,0,1)$. □

Aside: is V_{si} closed under unions of chains?

Corollary 2. *Every $\mathbf{A} \in V_{si}^{01}$ can be extended to a $\leq$-maximal member of V_{si}^{01}.*

Proof. Residual smallness, Proposition 1, and Zorn's Lemma. □

Definition. Suppose $\mathbf{A}$ is an algebra in signature σ, $C \subseteq A$, and $n \geq 1$.

(1) σ_C is the expansion of σ including each element of C as a new constant.
(2) $\mathbf{A}_C$ is the obvious expansion of $\mathbf{A}$ to the signature σ_C.
(3) $\mathcal{F}_{pp}(n, C)$ is the set of all pp-formulas φ in the signature σ_C and free variables $x_1, \ldots, x_n$.
(4) For $\mathbf{a} \in A^n$, $\mathrm{tp}_{pp}(\mathbf{a}/C) = \{\varphi(\mathbf{x}) \in \mathcal{F}_{pp}(n,C) : \mathbf{A}_C \models \varphi(\mathbf{a})\}$.

Example. *Suppose $\mathbf{A} = (\mathbb{R}, +, -, \times, 0, 1)$, $C = \mathbb{Q}$, $a = \pi$. $\mathrm{tp}_{pp}(\pi/\mathbb{Q})$ contains formulas $\varphi(x)$ such as*

$$\begin{aligned} &\exists y(3.14 + y^2 = x) \\ &\exists y(3.141 + y^2 = x) \\ &\vdots \\ &\exists y(x + y^2 = 3.15) \\ &\exists y(x + y^2 = 3.142) \\ &\vdots \end{aligned}$$

Lecture 2, April 13, 2018

Recall:
σ is a countable signature containing constants $0, 1$.
V is a residually small variety in signature σ.
Σ is a set of axioms for V.
$V_{si} = \{\mathbf{A} \in V : \mathbf{A} \text{ is SI}\}$.
$V_{si}^{01} := \{\mathbf{A} \in V_{si} : (0^{\mathbf{A}}, 1^{\mathbf{A}}) \in \mu_{\mathbf{A}} \setminus 0_A\}$.

- Assumption: $\{|A| : \mathbf{A} \in V_{si}\} = \{|A| : \mathbf{A} \in V_{si}^{01}\}$.

Fact: Every $\mathbf{A} \in V_{si}$ has size $|A| \leq 2^{\omega}$.
Corollary 2: every $\mathbf{A} \in V_{si}^{01}$ can be extended to a maximal member of V_{si}^{01}.
Goal: if $\exists \mathbf{A} \in V_{si}^{01}$ with $|A| > \omega$, then $\exists \mathbf{A} \in V_{si}^{01}$ with $|A| = 2^{\omega}$.

Definition. Suppose $\mathbf{A} \in V$ and $C \subseteq A$.

(1) $\sigma_C = \sigma \cup C$.
(2) $\mathcal{F}_{pp}(1, C)$ is the set of all pp-formulas $\varphi(x)$ in the signature σ_C.
(3) For $a \in A$, $\mathrm{tp}_{pp}(a/C) = \{\varphi(x) \in \mathcal{F}_{pp}(1, C) : \mathbf{A}_C \models \varphi(a)\}$.

Lemma 3. *Suppose* $\mathbf{A}$ *is maximal member of* V_{si}^{01} *and* $\mathbf{A} \leq \mathbf{B} \in V$.

(1) *There exists a retraction* $r : \mathbf{B} \to \mathbf{A}$ *with* $r\restriction_A = \mathrm{id}_A$.
(2) *Hence for all* $C \subseteq A$ *and* $b \in B$ *there exists* $a \in A$ *with* $\mathrm{tp}_{pp}^{\mathbf{B}}(b/C) \subseteq \mathrm{tp}_{pp}^{\mathbf{A}}(a/C)$.

Proof. (1) Let $0 = 0^{\mathbf{A}}$ $(= 0^{\mathbf{B}})$ and similarly for 1.
Pick $\theta \in \mathrm{Con}\,\mathbf{B}$ maximal w.r.t. $(0, 1) \notin \theta$.
Let $\overline{\mathbf{B}} = \mathbf{B}/\theta$. Clearly $\overline{\mathbf{B}}$ is SI, $0^{\overline{\mathbf{B}}} \neq 1^{\overline{\mathbf{B}}}$, and $(0^{\overline{\mathbf{B}}}, 1^{\overline{\mathbf{B}}}) \in \mu_{\overline{\mathbf{B}}}$. So $\overline{\mathbf{B}} \in V_{si}^{01}$.
Let $h : \mathbf{A} \to \overline{\mathbf{B}}$ be $h = \nu_\theta \circ incl$. (Draw diagram.)
Clearly $h(0) \neq h(1)$. So $(0, 1) \notin \ker(h)$. I.e., $\mu_{\mathbf{A}} \not\leq \ker(h)$.
So $\ker(h) = 0_A$, i.e., h is injective.
So $h : \mathbf{A} \hookrightarrow \overline{\mathbf{B}} \in V_{si}^{01}$. But $\mathbf{A}$ is a maximal member of V_{si}^{01}. Thus h is surjective, i.e., $h : \mathbf{A} \cong \overline{\mathbf{B}}$. (Add h^{-1} to diagram.)
Let $r = h^{-1} \circ \nu_\theta : \mathbf{B} \to \mathbf{A}$.
Check that for $a \in A$,

$$h(a) = a/\theta, \quad \text{so } r(a) = h^{-1}(a/\theta) = a.$$

(2) Let r be the retraction from (1).
Fix $b \in B$. Let $a = r(b)$. I claim that $\mathrm{tp}_{pp}^{\mathbf{B}}(b/C) \subseteq \mathrm{tp}_{pp}^{\mathbf{A}}(a/C)$.
Assume $\varphi(x) \in \mathrm{tp}_{pp}^{\mathbf{B}}(b/C)$. So $\mathbf{B}_C \models \varphi(b)$.
Note that $r : \mathbf{B}_C \to \mathbf{A}_C$.
Surjective homomorphisms preserve positive formulas, so $\mathbf{A}_C \models \varphi(r(b))$.
I.e., $\mathbf{A}_C \models \varphi(a)$, so $\varphi(x) \in \mathrm{tp}_{pp}^{\mathbf{A}}(a/C)$. □

Definition. Suppose $\mathbf{A}$ is an algebra in signature σ. $\mathrm{Diag}^+\mathbf{A}$ is the set of atomic σ_A-sentences true in $\mathbf{A}_A$.

Lemma 4. *Suppose* $\mathbf{A}$ *is a maximal member of* V_{si}^{01}*,* $C \subseteq A$*, and* $\Phi \subseteq \mathcal{F}_{pp}(1, C)$*. TFAE:*

(1) *There exists* $a \in A$ *with* $\Phi \subseteq \mathrm{tp}_{pp}(a/C)$*.*
(2) $\Phi \cup \underbrace{\mathrm{Diag}^+\mathbf{A} \cup \Sigma \cup \{0 \neq 1\}}_{T}$ *is consistent.*

Proof. (1) $\Rightarrow$ (2). If $\Phi \subseteq \mathrm{tp}_{pp}(a/C)$, then $\mathbf{A}_A \models T$, and $\mathbf{A}_A \models \varphi(a)$ for all $\varphi(x) \in \Phi$. So $(\mathbf{A}_A, a)$ is a model of $\Phi \cup T$.

(2) $\Rightarrow$ (1). Assuming (2), there exists an algebra $\mathbf{B}^\circ$ in signature σ_A with $\mathbf{B}^\circ \models T$, and an element $b^* \in B$ satisfying $\mathbf{B}^\circ \models \varphi(b^*)$ for all $\varphi(x) \in \Phi$.

Let $\mathbf{B} =$ the reduct of $\mathbf{B}^\circ$ to σ.

For $a \in A$ let $b_a := a^{\mathbf{B}^\circ}$.

Thus we can write $\mathbf{B}^\circ = (\mathbf{B}, (b_a : a \in A))$.

Define $\mathbf{B}_{\text{“}C\text{”}} = (\mathbf{B}, (b_c : c \in C))$, the reduct of $\mathbf{B}^\circ$ to signature σ_C.

Define $h : A \to B$ by $h(a) = b_a$.

Claim 1. $\mathbf{B} \in V$, $0^{\mathbf{B}} \neq 1^{\mathbf{B}}$, and $\mathbf{B}_{\text{“}C\text{”}} \models \varphi(b^*)$ for all $\varphi(x) \in \Phi$. □

Claim 2. h is a homomorphism $\mathbf{A}_C \to \mathbf{B}_{\text{“}C\text{”}}$.

Proof. Let f be an n-ary fundamental operation symbol in σ, let $a_1, \ldots, a_n \in A$, and let $a = f^{\mathbf{A}}(a_1, \ldots, a_n)$. Then the sentence $f(a_1, \ldots, a_n) = a$ is in $\mathrm{Diag}^+\mathbf{A}$, so $\mathbf{B}^\circ \models f(a_1, \ldots, a_n) = a$, which means

$$f^{\mathbf{B}^\circ}(b_{a_1}, \ldots, b_{a_n}) = b_a$$

and hence

$$f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = h(a) = h(f^{\mathbf{A}}(a_1, \ldots, a_n)).$$

Next, let $c \in C$. Obviously $h(c^{\mathbf{A}_C}) = h(c) = b_c = c^{\mathbf{B}_{\text{“}C\text{”}}}$. □

Claim 3. h is injective.

Proof. $h(0) = h(0^{\mathbf{A}}) = 0^{\mathbf{B}} \neq 1^{\mathbf{B}} = h(1^{\mathbf{A}}) = h(1)$, so $(0, 1) \notin \ker(h)$. □

Hence $\mathbf{A}_C \hookrightarrow \mathbf{B}_{\text{“}C\text{”}}$. So WLOG we can assume $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B}_{\text{“}C\text{”}} = \mathbf{B}_C$.

So $\Phi \subseteq \mathrm{tp}^{\mathbf{B}}_{pp}(b^*/C)$, by Claim 1.

By Lemma 3(2), there exists $a \in A$ with $\mathrm{tp}^{\mathbf{B}}_{pp}(b^*/C) \subseteq \mathrm{tp}^{\mathbf{A}}_{pp}(a/C)$. □