Lecture 5, May 18, 2018

Recall:

V is a residually small variety in signature σ.

$V_{si}^{01} := \{\mathbf{A} \in V_{si} : (0^{\mathbf{A}}, 1^{\mathbf{A}}) \in \mu_{\mathbf{A}} \backslash 0_A\}$.

Corollary 2: every $\mathbf{A} \in V_{si}^{01}$ can be extended to a maximal member of V_{si}^{01}.

If $\mathbf{A}$ is maximal in V_{si}^{01} and $C \subseteq A$ with $|C| \leqslant \omega$, then $S(C) = \{\mathrm{tp}_{pp}(a/C) : a \in A\}$.

Corollary 8. For such $\mathbf{A}$ and C, either $|S(C)| \leqslant \omega$ or $|S(C)| = 2^\omega$.

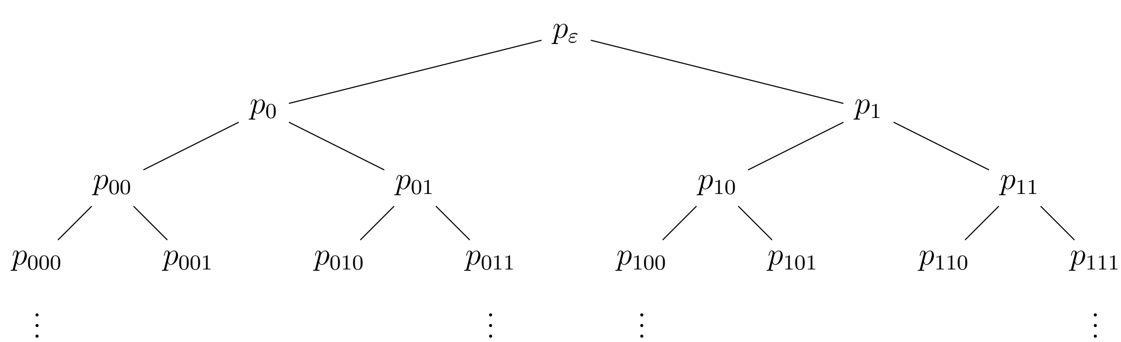
Theorem 9. *If $\mathbf{A} \in V_{si}^{01}$ is maximal and $|A| > \omega$, then $|A| = 2^\omega$.*

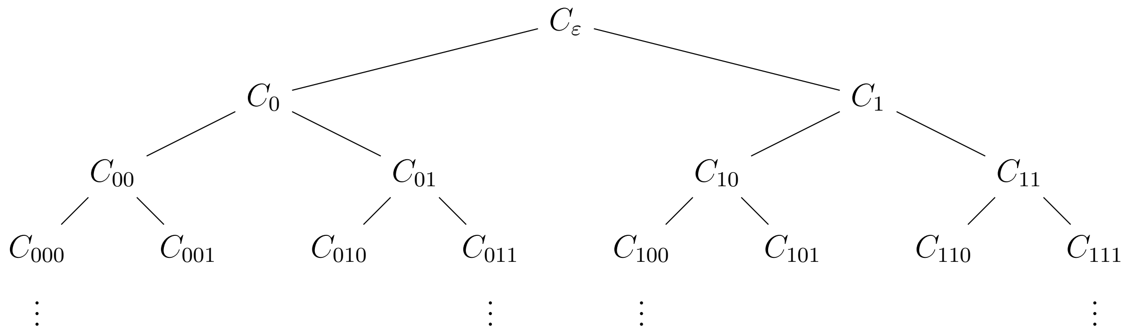
Proof. We just need to prove $|A| \geqslant 2^\omega$.

Case 1: There exists $C \subseteq A$ with $|A| \leqslant \omega$ and $|S(C)| > \omega$.
Then $|A| \geqslant |S(C)| = 2^\omega$ by Corollary 8.

Case 2: Otherwise. (For all $C \subseteq A$, $|C| \leqslant \omega \implies |S(C)| \leqslant \omega$)
For $C \subseteq A$ with $|C| \leqslant \omega$ and $p \in S(C)$, let $[\![p]\!] = \{a \in A : \mathrm{tp}_{pp}(a/C) = p\}$.
The sets $[\![p]\!]$ $(p \in S(C))$ partition A into countably many classes, at least one of which is uncountable.
Define $S_1(C) = \{p \in S(C) : |[\![p]\!]| > \omega\}$.
Also note that if $C \subseteq D \subseteq A$ and $p \in S(C)$, then there exists $q \in S(D)$ with $p \subseteq q$.
If $p \in S_1(C)$, then there exists $q \in S_1(D)$ with $p \subseteq q$.

Subcase 2a: For all $C \subseteq A$ with $|C| \leqslant \omega$ and all $p \in S_1(C)$, there exists D with $C \subseteq D \subseteq A$ and $|D| \leqslant \omega$, and there exist $q_1, q_2 \in S_1(D)$ with $p \subseteq q_1, q_2$ and $q_1 \neq q_2$.
In this case we can define a binary tree of types over increasing countable sets.
Start with any $C_\varepsilon \subseteq A$ with $|C_\varepsilon| \leqslant \omega$ and $p_\varepsilon \in S_1(C_\varepsilon)$.





At each node we have $C_\eta \subseteq C_{\eta 0} \cap C_{\eta 1}$ and $p_\eta \subseteq p_{\eta 0} \cap p_{\eta 1}$ and $p_{\eta 0} \neq p_{\eta 1}$.

There are countably many sets C_η in the tree. Let $C = \bigcup_\eta C_\eta$ and note $|C| \leqslant \omega$.

If we take the union of the p_η's down any branch, we get a subset of $\mathcal{F}_{pp}(1, C)$ which is consistent with $\Sigma \cup \mathrm{Diag}^+ \mathbf{A} \cup \{0 \neq 1\}$, hence which can be extended to a type over C.

Different branches must lead to different types (exercise). Hence $|A| \geqslant 2^\omega$.

Subcase 2b: Else.

$\exists\ C \subseteq A$ with $|C| \leqslant \omega$, $\exists\ p \in S_1(C)$, such that for all D with $C \subseteq D \subseteq A$ and $|D| \leqslant \omega$, there is a unique $q \in S_1(D)$ with $p \subseteq q$. Fix such C and p.

Lemma. *Suppose $C \subseteq D_1 \subseteq D_2 \subseteq A$ with $|D_2| \leqslant \omega$, and $q_i \in S_1(D_i)$ is the unique extension of p for $i = 1, 2$. Then $q_1 \subseteq q_2$.*

Proof. Pick $b \in [\![q_2]\!]$ and let $q_1' = \mathrm{tp}_{pp}(b/D_1)$. Clearly $p \subseteq q_1' \subseteq q_2$, so $[\![q_2]\!] \subseteq [\![q_1']\!]$, so $[\![q_1']\!]$ is uncountable, meaning $q_1' \in S_1(D_1)$. By uniqueness, $q_1' = q_1$. □

Now we define by transfinite recursion three sequences (C_α), (a_α), (q_α) $(\alpha < \omega_1)$ as follows:

$$\begin{aligned} C_\alpha &= C \cup \{a_\beta : \beta < \alpha\} \\ q_\alpha &= \text{the unique extension of } p \text{ in } S_1(C_\alpha) \\ a_\alpha &\in [\![q_\alpha]\!] \text{ (any).} \end{aligned}$$

We have $C_0 \subseteq C_1 \subseteq \cdots$ and $p = q_0 \subseteq q_1 \subseteq \cdots$ (the latter by the Lemma). Thus $[\![p]\!] = [\![q_0]\!] \supseteq [\![q_1]\!] \supseteq \cdots$ is an uncountable decreasing sequence with each set uncountable. (In effect, our subcase guarantees that the intersection of any countable number of them is always uncountable, so we can always keep going.)

Can show that $a_\alpha \notin C_\alpha$ (else $[\![q_\alpha]\!] = \{c_\alpha\}$), so the elements $a_0, a_1, \ldots$ are pairwise distinct.

Thus for each $\alpha < \omega_1$ we have $(0, 1) \in \mathrm{Cg}(a_\alpha, a_{\alpha+1})$. Pick a principal congruence formula π_α in signature σ witnessing this.

There are only countable many principal congruence formulas, so there is an uncountable subset $S \subseteq \omega_1$ and a single principal congruence formula π such that $\pi_\alpha = \pi$ for all $\alpha \in S$.

Claim: for all $\alpha, \beta \in S$ with $\alpha < \beta$, $\mathbf{A} \models \pi(a_\alpha, a_\beta, 0, 1)$.

Proof of claim. $\alpha < \beta$ implies $q_{\alpha+1} \subseteq q_\beta$, so $[\![q_{\alpha+1}]\!] \supseteq [\![q_\beta]\!]$, so $a_\beta \in [\![q_{\alpha+1}]\!]$, which means

$$\mathrm{tp}_{pp}(a_\beta / C_{\alpha+1}) = q_{\alpha+1} = \mathrm{tp}_{pp}(a_{\alpha+1} / C_{\alpha+1}).$$

Since $a_\alpha \in C_{\alpha+1}$ we have $\pi(a_\alpha, x, 0, 1) \in q_{\alpha+1}$, so $\mathbf{A} \models \pi(a_\alpha, a_\beta, 0, 1)$. $\square$

But this is a big problem: as Justin explained on March 16, in a residually small variety you cannot have the above Claim holding with S infinite. $\square$