Lecture 3, May 4, 2018

Recall:
σ is a countable signature containing constants $0, 1$.
V is a residually small variety in signature σ.
Σ is a set of axioms for V.
$V_{si} = \{\mathbf{A} \in V : \mathbf{A} \text{ is SI}\}$.
$V_{si}^{01} := \{\mathbf{A} \in V_{si} : (0^{\mathbf{A}}, 1^{\mathbf{A}}) \in \mu_{\mathbf{A}} \setminus 0_A\}$.
Fact: Every $\mathbf{A} \in V_{si}$ has size $|A| \leq 2^\omega$.
Corollary 2: every $\mathbf{A} \in V_{si}^{01}$ can be extended to a maximal member of V_{si}^{01}.
Goal: if $\exists \mathbf{A} \in V_{si}^{01}$ with $|A| > \omega$, then $\exists \mathbf{A} \in V_{si}^{01}$ with $|A| = 2^\omega$.

Definition. Suppose $\mathbf{A} \in V$ and $C \subseteq A$.

(1) $\sigma_C = \sigma \cup C$. $\mathbf{A}_C$ is the natural expansion of $\mathbf{A}$ to σ_C.
(2) $\mathcal{F}_{pp}(n, C)$ is the set of all pp-formulas $\varphi(x_1, \ldots, x_n)$ in the signature σ_C.
(3) For $\mathbf{a} \in A^n$, $\mathrm{tp}_{pp}(\mathbf{a}/C) = \{\varphi(\mathbf{x}) \in \mathcal{F}_{pp}(n, C) : \mathbf{A}_C \models \varphi(\mathbf{a})\}$.
(4) $\mathrm{Diag}^+\mathbf{A}$ is the set of atomic σ_A-sentences true in $\mathbf{A}_A$.

Lemma 4. Suppose $\mathbf{A}$ is a maximal member of V_{si}^{01}, $C \subseteq A$, and $\Phi \subseteq \mathcal{F}_{pp}(1, C)$. TFAE:

(1) There exists $a \in A$ with $\Phi \subseteq \mathrm{tp}_{pp}(a/C)$.
(2) $\Phi \cup \mathrm{Diag}^+\mathbf{A} \cup \Sigma \cup \{0 \neq 1\}$ is consistent.

Lemma 5. *Suppose* $\mathbf{A}$ *is maximal in* V_{si}^{01} *and* $C \cup \{a, b\} \subseteq A$. *TFAE:*

(1) $\mathrm{tp}_{pp}(a/C) \subseteq \mathrm{tp}_{pp}(b/C)$.
(2) *There exists* $h : \mathbf{A}_C \to \mathbf{A}_C$ *with* $h(a) = b$.
(3) *There exists* $h \in \mathrm{Aut}\,\mathbf{A}_C$ *with* $h(a) = b$.
(4) $\mathrm{tp}_{pp}(a/C) = \mathrm{tp}_{pp}(b/C)$.

Proof. Clearly (3) $\Rightarrow$ (4) $\Rightarrow$ (1).
(2) $\Rightarrow$ (3) is also easy. Let $h : \mathbf{A}_C \to \mathbf{A}_C$ be given.
We have $h(0) \neq h(1)$, so h is an embedding $\mathbf{A} \hookrightarrow \mathbf{A}$.
The image $h(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{A}$. But $\mathbf{A}$ is maximal in V_{si}^{01}, so the image must equal $\mathbf{A}$, i.e., h is surjective. Hence $h \in \mathrm{Aut}\,\mathbf{A}_C$.
So suffices to prove (1) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2). Assume $\mathrm{tp}_{pp}(a/C) \subseteq \mathrm{tp}_{pp}(b/C)$. We'll construct h as follows.
Let H be the set of all partial functions $h : \mathrm{dom}(h) \to A$ where

(i) $\mathrm{dom}(h) \subseteq A$.
(ii) $a \in \mathrm{dom}(h)$ and $h(a) = b$.

(*iii*) For all $n \geq 1$ and $\mathbf{a} \in \mathrm{dom}(h)^n$,

$$\mathrm{tp}_{pp}(\mathbf{a}/C) \subseteq \mathrm{tp}_{pp}(h(\mathbf{a})/C).$$

By assumption, the function h with $\mathrm{dom}(h) = \{a\}$ and $h(a) = b$ is in H. So $H \neq \varnothing$.

Also, H is closed under unions of chains (exercise).

So H has a maximal element h. Let $D = \mathrm{dom}(h)$. If we can show that $D = A$, then $h : A \to A$, $h(a) = b$, and property (*iii*) above will give that h is a homomorphism $\mathbf{A}_C \to \mathbf{A}_C$ (exercise).

So suffices to show $D = A$. Suppose $D \neq A$. Pick any $a^* \in A \setminus D$. Let

$$\Phi = \{\varphi(x, h(\mathbf{a})) \,:\, n \geq 0,\ \varphi \in \mathcal{F}_{pp}(n+1, C),\ \mathbf{a} \in D^n,\ \mathbf{A}_C \models \varphi(a^*, \mathbf{a})\}$$

and note that $\Phi \subseteq \mathcal{F}_{pp}(1, C \cup \mathrm{ran}(h))$.

Claim: $\Phi \cup \mathrm{Diag}^+\mathbf{A} \cup \Sigma \cup \{0 \neq 1\}$ is consistent.

Proof. By compactness and the fact that Φ is closed under conjunction, it suffices to show that

$$\{\varphi(x, h(\mathbf{a}))\} \cup \Sigma \cup \mathrm{Diag}^+\mathbf{A} \cup \{0 \neq 1\}$$

is consistent for each $\varphi(x, h(\mathbf{a})) \in \Phi$. We have $\varphi \in \mathcal{F}_{pp}(n+1, C)$, $\mathbf{a} \in D^n$ and $\mathbf{A}_C \models \varphi(a^*, \mathbf{a})$. Let $\psi(\mathbf{x}) = \exists y \varphi(y, \mathbf{x}) \in \mathcal{F}_{pp}(n, C)$.

$$\begin{aligned}
\mathbf{A}_C \models \varphi(a^*, \mathbf{a}) &\implies \mathbf{A}_C \models \psi(\mathbf{a}) \\
&\implies \psi \in \mathrm{tp}_{pp}(\mathbf{a}/C) \subseteq \mathrm{tp}_{pp}(h(\mathbf{a})/C) \quad \text{by } (iii) \\
&\implies \mathbf{A}_C \models \psi(h(\mathbf{a})) \\
&\implies \mathbf{A}_A \models \{\exists y \varphi(y, h(\mathbf{a}))\} \cup \Sigma \cup \mathrm{Diag}^+\mathbf{A} \cup \{0 \neq 1\},
\end{aligned}$$

which shows that $\{\varphi(x, h(\mathbf{a}))\} \cup \Sigma \cup \mathrm{Diag}^+\mathbf{A} \cup \{0 \neq 1\}$ is consistent. □

Hence by Lemma 4, there exists $b^* \in A$ with $\Phi \subseteq \mathrm{tp}_{pp}(b^*/C \cup \mathrm{ran}(h))$.

Hence for any $\mathbf{a} \in D^n$,

for any $\varphi \in \mathcal{F}_{pp}(n+1, C)$,

$$\begin{aligned}
\varphi \in \mathrm{tp}_{pp}((a^*, \mathbf{a})/C) &\implies \mathbf{A}_C \models \varphi(a^*, \mathbf{a}) \\
&\implies \varphi(x, h(\mathbf{a})) \in \Phi \\
&\implies \varphi(x, h(\mathbf{a})) \in \mathrm{tp}_{pp}(b^*/C \cup \mathrm{ran}(h)) \\
&\implies \mathbf{A}_C \models \varphi(b^*, h(\mathbf{a})),
\end{aligned}$$

which proves $\mathrm{tp}_{pp}((a^*, \mathbf{a})/C) \subseteq \mathrm{tp}_{pp}((b^*, h(\mathbf{a}))/C)$.

Extend h to h^* with $\mathrm{dom}(h^*) = D \cup \{a^*\}$ by setting $h^*(a^*) = b^*$.

Last sequence of $\implies$'s essentially say that h^* satisfies (*iii*).

So $h^* \in H$, contradicting maximality of h. □