Oct 3

Let $\mathbf{M} = (M, \mathcal{F})$ be a finite algebra. (Think $\mathbf{2}$ of $\mathbf{2}_{BDL}$.) Thus $M \neq \varnothing$, $|M| < \omega$, and $\mathcal{F} = \{f^{\mathbf{M}} : f \in \sigma\}$ is a set of finitary operations on M. σ is the *signature*. Ω_σ is the class of all algebras of signature σ.

Aim: to provide a Stone/Priestley-like duality for $\mathrm{ISP}(\mathbf{M}) = \mathrm{Sep}(\Omega_\sigma, \mathbf{M})$.

We need a discrete topological structure $\mathbb{M}$ (analogous to $\mathbb{2}$ or $\mathbb{2}_{pos}$). It will have the same universe as $\mathbf{M}$ and also have

- A set $\mathcal{G} = \{g^{\mathbb{M}} : g \in G\}$ of finitary operations on M indexed by a set G.
- A set $\mathcal{H} = \{h^{\mathbb{M}} : h \in H\}$ of partial finitary operations on M indexed by H.
- A set $\mathcal{R} = \{r^{\mathbb{M}} : r \in R\}$ of finitary relations on M indexed by R.
- The discrete topology.

$\tau = (G, H, R)$ is the *signature* of $\mathbb{M}$.

Conventions:

- G can include 0-ary ops (constants). Arities of $H \cup R$ must be > 0.
- Domains of partial operations in $\mathcal{H}$, and relations in $\mathcal{R}$, must be nonempty.

Definition 1.4. Given a nonempty set I, $\mathbb{M}^I$ is a topological structure of signature τ defined as follows:

1. Universe: the set M^I of all functions $x : I \to M$.
2. Operations $g^{\mathbb{M}^I} : (M^I)^n \to M^I$ defined from $g^{\mathbb{M}}$ coordinatewise:
$$\left(g^{\mathbb{M}^I}(x_1, \ldots, x_n)\right)(i) = g^{\mathbb{M}}(x_1(i), \ldots, x_n(i)).$$
3. Partial operations defined likewise, with
$$\mathrm{dom}(h^{\mathbb{M}^I}) = \{(x_1, \ldots, x_n) : (x_1(i), \ldots, x_n(i)) \in \mathrm{dom}(h^{\mathbb{M}})\}.$$
4. Relations $r^{\mathbb{M}^I}$ defined coordinatewise (like domains of partial operations).
5. Topology: $U \subseteq M^I$ is open iff $\forall x \in U$ there exists a finite subset $F \subseteq I$ such that $\forall y \in M^I$, $y|_F = x|_F$ implies $y \in U$.

Facts. With this topology:

1. M^I topology is a Stone space.
2. Each operation $g^{\mathbb{M}^I}$ is continuous.
3. Each n-ary partial operation $h^{\mathbb{M}^I}$ is such that its domain $\mathrm{dom}(h^{\mathbb{M}^I})$ is a closed in $(\mathbb{M}^I)^n$, and is continuous as a function $\mathrm{dom}(h^{\mathbb{M}^I}) \to M^I$.
4. Each n-ary relation $r^{\mathbb{M}^I}$ is a closed subset of $(M^I)^n$.

Definition 1.5. Given $\mathbb{M}$, $I \neq \varnothing$, and $X \subseteq M^I$,

1. X is a **subuniverse** of $\mathbb{M}^I$ if
   (a) For all $g \in G$ of arity n, if $x_1, \ldots, x_n \in X$ then $g^{\mathbb{M}^I}(x_1, \ldots, x_n) \in X$.

(b) For all $h \in H$ of arity n, if $x_1, \ldots, x_n \in X$ and $(x_1, \ldots, x_n) \in \mathrm{dom}(h^{\mathbb{M}^I})$, then $h^{\mathbb{M}^I}(x_1, \ldots, x_n) \in X$.

(2) If X is a subuniverse of $\mathbb{M}^I$, then the induced **substructure** of $\mathbb{M}^I$ is the topological structure $\mathbb{X}$ in signature τ with universe X and operations, partial operations, relations, and topology induced by $\mathbb{M}^I$.

Definition 1.6. Given $\mathbf{A} \in \Omega_\sigma$, $\mathrm{Hom}(\mathbf{A}, \mathbf{M}) = \{$all homomorphisms $f : \mathbf{A} \to \mathbf{M}\}$. Given $\mathbb{X} \in \Omega_\tau^{top}$, $\mathrm{Hom}(\mathbb{X}, \mathbb{M}) = \{$all continuous homomorphisms $\varphi : \mathbb{X} \to \mathbb{M}\}$.

What we want: for all $\mathbf{A} \in \mathrm{ISP}(\mathbf{M})$,

(D0) $X := \mathrm{Hom}(\mathbf{A}, \mathbf{M})$ is closed in M^A. (This is always true!)

(D1) X is a subuniverse of $\mathbb{M}^A$. (Gives $\mathbb{X} \in \Omega_\tau^{Stone}$.)

(D2) $E(\mathbb{X}) := \mathrm{Hom}(\mathbb{X}, \mathbb{M})$ is a subalgebra of $\mathbf{M}^X$. (Gives $\mathbf{E}(\mathbb{X}) \in \Omega_\sigma$.)

(D3) $e_{\mathbf{A}} : A \to E(\mathbb{X})$ given by

$$(e_{\mathbf{A}}(a))(x) = x(a)$$

is an isomorphism $\mathbf{A} \cong \mathbf{E}(\mathbb{X})$.

Definition 1.7. When (D1)–(D3) hold for all $\mathbf{A} \in \mathrm{ISP}(\mathbf{M})$, we say that $\mathbb{M}$ **yields a duality on** $\mathrm{ISP}(\mathbf{M})$ (or **dualizes M**).

Aim for this lecture: to simplify conditions (D1)–(D3).

Definition 1.8. Fix $m, n > 0$. Let f be an n-ary operation on M, g an m-ary operation or partial operation on M, m an n-ary relation on M, and $c \in M$.

- $M^{m \times n}$ is the set of all $m \times n$ matrices with entries from M.
- g **commutes with** f if for all $(e_{ij}) \in M^{m \times n}$, if $\mathrm{col}_1, \ldots, \mathrm{col}_n \in \mathrm{dom}(g)$ then $(f(\mathrm{row}_1), \ldots, f(\mathrm{row}_m)) \in \mathrm{dom}(g)$ and
$$f(g(\mathrm{col}_1), \ldots, g(\mathrm{col}_n)) = g(f(\mathrm{row}_1), \ldots, f(\mathrm{row}_m)).$$
- g **commutes with** c if $(c, c, \ldots, c) \in \mathrm{dom}(g)$ and $g(c, c, \ldots, c) = c$.
- r is **invariant under** f if for all $(e_{ij}) \in M^{m \times n}$, if $\mathrm{col}_1, \ldots, \mathrm{col}_n \in r$ then
$$(f(\mathrm{row}_1), \ldots, f(\mathrm{row}_m)) \in r.$$
- r is **invariant under** c if $(c, c, \ldots, c) \in r$.