Brute Force

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Preliminaries

Last time, we got a necessary condition for a duality: that the data of $\underset{\sim}{\mathbf{M}} = (M, G, H, R, \mathcal{T})$ be invariant with respect to the data of $\underline{\mathbf{M}} = (M, F)$.

We didn't prove it, but this invariance criterion is equivalent to the graphs of the operations f in $\underline{\mathbf{M}}$ being closed subalgebras of the appropriate powers of $\underset{\sim}{\mathbf{M}}$, and to the graphs of the operations h and relations r in $\underset{\sim}{\mathbf{M}}$ being subalgebras of powers of $\underline{\mathbf{M}}$. This is useful conceptually, and will be useful in the immediate future.

For the foreseeable future, we will assume that every $\underset{\sim}{\mathbf{M}}$ has these properties.

Intro

One observation: if (G, H, R) yields a duality, this means exactly that the *only* continuous homomorphisms $\underset{\sim}{\mathbf{D}}(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}}) \to \underset{\sim}{\mathbf{M}}$ are the evaluation homomorphisms, that is, that

$$e_{\mathbf{A}} : \mathbf{A} \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}}), \underset{\sim}{\mathbf{M}}) = \mathbf{E}(\underset{\sim}{\mathbf{D}}(\mathbf{A}))$$

is surjective. (Remember that the injectivity of this map comes from the separation property of $\mathbf{A}$, and its homomorphism-ness comes from *pre*-duality.)

Given a candidate (G, H, R) which may or may not yield a duality, how can we modify it into some (G', H', R'), so that it's no worse at yielding a duality? Well, if $G \subseteq G'$, $H \subseteq H'$, and $R \subseteq R'$, then any map $\underset{\sim}{\mathbf{D}}(\mathbf{A}) \to \underline{\mathbf{M}}$ which preserves the latter structure also preserves the former. If (G, H, R) "weeds out" all of the non-evaluation homomorphisms $\underset{\sim}{\mathbf{D}}(\mathbf{A}) \to \underline{\mathbf{M}}$, then so too does (G', H', R'), so that (G', H', R') yields a duality if (G, H, R) does.

Explicitly, if all we seek is a duality, then there is nothing to be lost by taking as many operations and relations as we can. So, what is the space of these operations and relations from which we may greedily draw? Well, as I said earlier, the invariance criterion for h or r is nothing more and nothing less than being a subalgebra of a power of $\underline{\mathbf{M}}$.

Then, when the question "what should we choose G, H, and R to be?" presents itself, we might be tempted to answer: "why not take every subalgebra of every power of $\underline{\mathbf{M}}$?" This is the brute force approach, and it provides a sufficient answer to the problem of finding a duality. However, this previous response fails to specify how we should encode subalgebras $\mathbf{A} \leq \underline{\mathbf{M}}^{n+1}$: as operations $h : M^n \to M$ (if this is even possible) or as relations $R \subseteq M^{n+1}$.

Seeing as the former might not always work, while the latter always does, we'd like to work with "purely relational" structures. Let's observe that this actually works.

Brute Force

Lemma. *Let $\mathbf{A} \in \mathbb{ISP}\underline{\mathbf{M}}$, and suppose that $\underset{\sim}{\mathbf{M}} = (M, G, H, R)$ and $\underset{\sim}{\mathbf{M}}' = (M, R')$, where*

$$R' = R \cup \{\mathrm{graph}(h) : h \in G \cup H\}.$$

Then, $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{A}$ if and only if $\underset{\sim}{\mathbf{M}}'$ does.

Proof. Suppose that $h \in G \cup H$ is n-ary with domain $\mathrm{dom}(h)$, let S and T be non-empty sets, extend h to M^S and M^T in the usual way, and let $X \subseteq M^S$ and $Y \subseteq M^T$ be closed under h. We'll show that a map $\alpha : X \to Y$ preserves h if and only if it preserves the relation $r = \mathrm{graph}(h)$.

Let h^S and h^T be the extensions of h to M^S and M^T, respectively, and similarly let r^S and r^T be the extensions of r to these structures. Let $x_1, \ldots, x_n, x_{n+1} \in X$, and let the s-th component of these tuples be $x_i(s)$. Then, suppose that $\alpha : X \to Y$ preserves h, so that

$$\begin{aligned}
(x_1, \ldots, x_n, x_{n+1}) \in r^S &\Rightarrow \text{for every } s \in S, (x_1(s), \ldots, x_n(s), x_{n+1}(s)) \in r \\
&\Rightarrow (x_1(s), \ldots, x_n(s)) \in \mathrm{dom}(h) \text{ and } h(x_1(s), \ldots, x_n(s)) = x_{n+1}(s) \\
&\Rightarrow (x_1, \ldots, x_n) \in \mathrm{dom}(h^S) \text{ and } h^S(x_1, \ldots, x_n) = x_{n+1} \\
&\Rightarrow (\alpha(x_1), \ldots, \alpha(x_n)) \in \mathrm{dom}(h^T) \text{ and } h^T(\alpha(x_1), \ldots, \alpha(x_n)) = \alpha(x_{n+1}) \\
&\Rightarrow (\alpha(x_1), \ldots, \alpha(x_n), \alpha(x_{n+1})) \in r^T.
\end{aligned}$$

The converse is similar. Thus, the set of continuous homomorphisms $\underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{Y}}$ is the same in the case of either signature, and in particular, the set of continuous homomorphisms $\underset{\sim}{\mathbf{D}}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ is the same in both cases. □

So, adding more to (G, H, R) never hurts, invariance is equivalent to sub-algebra-ness, and it suffices to look at (purely) relational structures. We are thus led to the titular construction.
Definition. The set of all finitary invariant relations on a finite algebra $\underline{\mathbf{M}} = (M, F)$ is given by the *brute force* set of relations

$$\mathcal{B} = \bigcup_{n \geq 1} \mathbb{S}(\underline{\mathbf{M}}^n),$$

the set of all subalgebras of powers of $\underline{\mathbf{M}}$.

Summarizing what we know so far, we have that
Lemma. *For an algebra* $\mathbf{A} \in \mathbb{ISP}\underline{\mathbf{M}}$*, the following are equivalent:*

(i) *there is some structure* $\underset{\sim}{\mathbf{M}} = (M, G, H, R, \mathcal{T})$ *which yields a duality on* $\mathbf{A}$*,*
(ii) *there is some relational structure* $\underset{\sim}{\mathbf{M}} = (M, R, \mathcal{T})$ *which yields a duality on* $\mathbf{A}$*,*
(iii) *the brute force structure* $\underset{\sim}{\mathbf{M}} = (M, \mathcal{B}, \mathcal{T})$ *yields a duality on* $\mathbf{A}$*,*
(iv) *the only continuous homomorphisms* $\underset{\sim}{\mathbf{D}}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *are the evaluation homomorphisms* $e_{\mathbf{A}}(a)$*.*

So that's nice. But, rather obviously, we'd prefer to have a smaller, more tangible signature than $\mathcal{B}$. Alas.