Point-set topology with topics - Errata page

Basic general topology for graduate studies (Edition 2024)
by Robert André

The text has been completely revised, corrected and expanded on **December 13, 2025**. Questions from a student have incited me to re-insert a chapter entitled "The space of z-ultrafilters" which I had removed from an earlier version of the book. Many thanks to the anonymous reader who has recommended significant improvements to the chapter covering realcompact spaces and the chapter covering perfect functions. Readers can access a sampling of a few portions of the book that carry updated revisions at

`https://www.math.uwaterloo.ca/~randre/sets/revised.pdf`

`https://www.math.uwaterloo.ca/~randre/sets/chapter21.pdf`

`https://www.math.uwaterloo.ca/~randre/sets/chapter22.pdf`

`https://www.math.uwaterloo.ca/~randre/sets/chapter32.pdf`

`https://www.math.uwaterloo.ca/˜randre/sets/content.pdf`

The chapter on the Stone space has been expanded to include a detailed example. As always, thoughts, suggestions and comments are welcome and appreciated.

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Errata page

In section:

5.6 Topic: Spaces with countable bases.

Example 10. Suppose τ_S represents the *upper limit topology* on $\mathbb{R}$ (the Sorgenfrey line). Recall that, when $\mathbb{R}$ is equipped with this topology, $\mathscr{B} = \{(x, y] : x, y \in \mathbb{R}\}$ is an open base for $\mathbb{R}$. Show that $(\mathbb{R}, \tau_S)$ is first countable but not second countable.

Solution : The space, $(\mathbb{R}, \tau_S)$ is first countable: For each $x \in \mathbb{R}$,

$$\mathscr{B}_x = \{(x - 1/n, x] : n \in \mathbb{N}\}$$

is a countable neighborhood base at x. So $(\mathbb{R}, \tau_S)$ is first countable.

The space, $(\mathbb{R}, \tau_S)$ is not second countable: Suppose $\mathscr{B} = \{(a_n, b_n] : n \in \mathbb{N}\}$ be a countable set of basic elements. Then there exists $z \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $z \neq b_n$. Then, for any $b < z$, $(b, z]$ is not the union of a subfamily of $\mathscr{B}$. So $(\mathbb{R}, \tau_S)$ does not have a countable base for open set.

So $(\mathbb{R}, \tau_S)$ is not second countable. As required

In section:

5.8 Topic : Hereditary topological properties.

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Definition 5.12 A topological property, say P, of a space (S, τ_S) is said to be a *hereditary topological property* provided every subspace, (T, τ_T), of S also has P.

Example 15. Show that metrizability is a hereditary property.

Solution: Suppose (S, τ) is metrizable. Then there exists a metric ρ such that (S, τ) and (S, ρ) have the same open sets. Suppose $T \subseteq S$ has the subspace topology and $\rho_t : T \times T \to \mathbb{R}$ is the subspace metric on T. Then (T, τ_t) and (T, ρ_t) have the same open sets and so T is metrizable. So subspaces of metrizable spaces are metrizable.

"First countable" is another example of a hereditary topological property.

However, the reader may want to verify that, if S is separable and V is an *open* subspace of S, then V is separable. But, in general, separability is not a hereditary property.

We now show that the "second countable" property is hereditary.

Theorem 5.13 Suppose (S, τ_S) is a second countable topological space. Then any non-empty subspace of S is also second countable. So "second countable" is a hereditary property.

Proof: Suppose (S, τ_S) has a countable base $\mathscr{B} = \{B_i : i \in \mathbb{N}\}$. Suppose (T, τ) is a non-empty subspace of S. Let U be an open subset of T. Then there exists an open subset U^* of S such that $U = U^* \cap T$. Then there exists $N \subseteq \mathbb{N}$ such that $U^* = \cup\{B_i : i \in N\}$. Then $U = \cup\{B_i \cap T : i \in N\}$. So $\mathscr{B}_T = \{B_i \cap T : i \in \mathbb{N}\}$ is a countable basis of T. Hence T inherits the second countable property from its superset S.

"Separable metrizable" is hereditary.

It is immediately worth noting that the above results allow us to conclude that *subspaces of separable metrizable spaces are separable.* That is, the "separable metrizable" property is hereditary. To see this, simply note that if T is a subspace of the a separable metrizable space S then T is metrizable (by the above example). We claim that T is separable: From theorem 5.11 , the metrizable space, S, must be second countable. By theorem 5.13, the second countable property is hereditary. So T must be both metrizable and second countable. Since second countable spaces are separable (by theorem 5.10, then T is a both metrizable and separable.

So, for example, should one want to argue that the irrationals, $\mathbb{J}$ (equipped with the usual topology), forms a separable space, it suffices to justify that $\mathbb{R}$ is both metrizable and second countable.

In section:

5.10 Topic : Topologizing an ordinal set.

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A few facts about an ordinal space. When we say "ordinal space" we mean a set of ordinals with the topology generated by the described subbase $\mathscr{S}$. But the best way to memorize the topology of the ordinal space is to remember what the elements of its base for open sets look like. Remember that there are two types of ordinals: limit ordinals and successor (non-limit) ordinals. Every ordinal number, α, without exception, has an immediate successor, $\alpha + 1$, by definition. Some ordinals, γ, have an immediate predecessor, say β, provided

$$\gamma = \beta + 1$$

In this case, $\sup[0, \gamma) = \sup[0, \beta + 1) = \sup[0, \beta] = \beta$.

While some ordinals, μ don't have an immediate predecessor (limit ordinal). In this case,

$$\sup\{\delta : \delta < \mu\} = \sup[0, \mu) = \mu$$

So when we consider the intersection of two elements, $(\mu, \omega_\gamma]$ and $[0, \beta)$, of the subbase, $\mathscr{S}$, we get the open-ended interval (μ, β). At this point, we see only two possible cases for β:

– Case 1 : Suppose β has an immediate predecessor, say γ (because $\beta = \gamma + 1$). In this case, we can express the open base element, (μ, β), as the half-open interval, $(\mu, \gamma]$.

– Case 2 : Suppose β doesn't have an immediate predecessor. Then $\beta = \{\delta : \delta < \beta\}$. In this case, we must leave (μ, β) as is.

In section:

6.8 Topic: The Pasting lemma and a generalization.

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Note that, in the above theorem, if the members of the collection $\mathscr{F}$ are all *open subsets* (rather than all closed as hypothesized in the theorem) the family $\mathscr{F}$ need not be locally finite for the statement to hold true. That is...

> *Let S and T be topological spaces and $\{O_i : i \in I\}$ be a collection of open subsets of S which covers all of S. Let $f : S \to T$ be a function. Then, $f : S \to T$ is a continuous function on S if and only if the restriction, $f|_{O_i}$, of f to O_i is continuous for each $i \in I$.*

It is left as an easy exercise for the reader to verify that this holds true.

Corollary 6.19 . . .

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In section:

10.5 Topic: On C^*-embedded subsets of normal spaces.

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Theorem 10.9
Let S be a T_1 topological space. Let A and B be any pair of non-empty disjoint closed subsets of S. The following are equivalent.

(a) The space S is a normal space.

(b) Each closed subset of S is C^*-embedded in S.

(c) Each closed subset of S is C-embedded in S.

Proof: We are given that (S, τ_S) is normal.

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(b) ⇒ (a) Suppose each closed subset of S is C^*-embedded in S. Let A and B be disjoint closed subsets of S. Then $A \cup B$ is a closed subset of S. Define $f : A \cup B \to \mathbb{R}$ as $f[A] = \{0\}$ and $f[B] = \{1\}$. By hypothesis, f extends continuously to $f^* : S \to \mathbb{R}$. Then f^* separates A and B. So S is normal.

(c) ⇒ (b) Suppose U is C-embedded in S. Consider a function, $f \in C^*(U)$. We claim f is C^*-embedded in S. Then there exists a number, M, such that $f[M] \in [-M, M]$. Since U is C-embedded, then f extends to $f^* \in C(S)$. Define the function $g : S \to \mathbb{R}$ as

$$g = (f^* \vee -M) \wedge M$$

We see that g is both a continuous and bounded function on S such that $g|_U = f$. So f in $C^*(U)$ will extend to $g \in C^*(S)$. So whenever U is C-embedded in S, then it is also C^*-embedded in S.

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In section:

13.6 Other properties of filters.

Remove Theorem 13.11

In section:

14.3 Properties of compact subsets.

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Example 6. Suppose S and T are topological spaces. We know that projection maps on product spaces are open maps. (See theorem on page 124)

Show that, in the case where the space T is compact then the projection map, $\pi_1 : S \times T \to S$, is a closed map.

Solution : Let K be a closed subset of $S \times T$. We are required to show that $\pi_1[K]$ is closed. It then suffices to show that $S \backslash \pi_1[K]$ is open in S.

Let $u \in S \backslash \pi_1[K]$. Then $(\{u\} \times T) \cap K = \varnothing$. Since T is compact then we easily see that $\{u\} \times T$ is compact. Since K is closed in $S \times T$, for each $(u, x) \in \{u\} \times T$ there is an open neighborhood, $V_u^x \times U_x$, which does not meet K. In this way we obtain an open cover

$$\{V_u^x \times U_x : x \in T\}$$

of $\{u\} \times T$ which then has a finite subcover

$$\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$$

Then $\cap\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$ forms an open neighborhood of u which does not intersect $\pi_1[K]$. So $S \backslash \pi_1[K]$ is open in S, as claimed. We conclude that $\pi_1 : S \times T \to S$ is a closed projection map.

In section:

14.5 Topic : Completely normal spaces.

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Perfectly normal spaces are completely normal.

Suppose S is perfectly normal. We claim that S must then be completely normal. By definition of "perfectly normal", S is a normal space. Let T be a subspace of S. To show that S is completely normal, it suffices to show that T is a normal subspace.

Let F and G be disjoint closed subsets of T. Then $F = F^* \cap T$ and $G = G^* \cap T$ for some closed subsets, F^* and G^*, of S. Since S is perfectly normal, F^* and G^* are both G_δ's in S, then F and G are disjoint G_δ's in T. Then there exists disjoint open neighborhoods of F and G in T. For, if not, F and G would intersect. Then T is normal. We conclude that S, is completely normal.

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In the next example we have a characterization of a completely normal space.

Example 8. Show that S is completely normal if and only if, for any pair of subsets A and B such that $A \cap \mathrm{cl}_S B = \varnothing = \mathrm{cl}_S A \cap B$ there exist disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.[5]

Solution: ($\Leftarrow$) Let S be a topological space. Suppose that for any pair of subsets A and B such that $A \cap \mathrm{cl}_S B = \varnothing = \mathrm{cl}_S A \cap B$ there exist *disjoint* open subsets U and V such that $A \subseteq U$ and $B \subseteq V$. If C and D are disjoint closed subsets of S, then $C \cap \mathrm{cl}_S D = \varnothing = \mathrm{cl}_S C \cap D$. Then, by hypothesis, there exist *disjoint* open subsets U and V such that $C \subseteq U$ and $D \subseteq V$. So S is normal.

We now claim that S is completely normal. Let T be a subspace of S. It suffices to show that T is normal. Suppose F and K are disjoint non-empty closed subsets of T. Then, it suffices to show that F and K are contained in disjoint open sets of T. Let F^* and K^* be closed subsets of S such that $F = F^* \cap T$ and $K = K^* \cap T$. Consider $H = K^* \cap F^*$. Then

$$\mathrm{cl}_S K^* \cap (F^* \backslash H) = \varnothing = \mathrm{cl}_S F^* \cap (K^* \backslash H)$$

By hypothesis, there exists disjoint open neighborhoods U and V in S which contain $(F^* \backslash H)$ and $(K^* \backslash H)$, respectively. Then $U \cap T$ and $V \cap T$ separate F and K in T.

[5] This is an exercise question which appears in 15B of S. Willard's Topology.

So T is normal, so S is completely normal.

The direction ($\Rightarrow$) is left as an exercise.

A normal space which is not perfectly normal. The following example illustrates a completely normal space which is not perfectly normal.

Example 9. Suppose S is an uncountable set and p be a special point in S. We equip S with a topology, τ, defined as follows:

$$\tau = \{T \subseteq S : S\backslash T \text{ is finite or } p \notin T\}$$

Show that S is completely normal but not perfectly normal.

Solution : Given: The set, τ, dictates that any finite subset of S is closed or any subset which contains p is closed. So S is T_1.

We claim that S is completely normal.
Proof of claim: Suppose F and K are disjoint non-empty subsets of S such that both $F \cap \mathrm{cl}_S K$ and $\mathrm{cl}_S K \cap F$ are empty. To prove the claim, it suffices to show that F and K can be separated by disjoint open subsets of S (by the statement in the previous example).
Now, the special point, p, can belong to only one of F, K or $S\backslash(F \cup K)$ (since these three sets partition S).
Case 1. If $p \in S\backslash(F \cup K)$, then K, F are open. So F and K are their own open sets which separate themselves from each other.
Case 2. Suppose $p \in K$. Since $\mathrm{cl}_S F \cap K = \varnothing$, then $p \notin \mathrm{cl}_S F$, so $\mathrm{cl}_S F$ is clopen in S. Then $S\backslash \mathrm{cl}_S F$ and $\mathrm{cl}_S F$ are disjoint open subsets which separate F and K.
Case 3. Suppose $p \in F$. Since $\mathrm{cl}_S K \cap F = \varnothing$, then $p \notin \mathrm{cl}_S K$, so $\mathrm{cl}_S K$ is clopen in S. Then $S\backslash \mathrm{cl}_S K$ and $\mathrm{cl}_S K$ are disjoint open subsets which separate F and K.
Whichever the case F and K can be completely separated by disjoint open subsets of S. By the above example, S is completely normal, as claimed.

We claim that S is not perfectly normal. To prove this, we must produce a closed subset of S which is not a G_δ. Our candidate is the set $\{p\}$. Let $\{U_i : i \in \mathbb{N}\}$ be a countable family of sets in S each containing the point p. Then each U_i is closed in S and $p \in \cap\{U_i : i \in \mathbb{N}\}$. If each U_i is also open, $S\backslash U_i$ is finite. So each U_i is uncountable. Then $S\backslash \cap\{U_i : i \in \mathbb{N}\} = \cup\{S\backslash U_i : i \in \mathbb{N}\}$, a countable subset of S. So $\cap\{U_i : i \in \mathbb{N}\}$ is uncountable. So $\{p\} \neq \cap\{U_i : i \in \mathbb{N}\}$. So p is not a G_δ. So S is not perfectly normal, as claimed.

So we can confidently state that ...,

$$\textit{completely normal} \not\Rightarrow \textit{perfectly normal}$$

In section:

17.2 Example of a compact space which is not sequentially compact.

Example 1. Let $S = [0,1]^{[0,1]}$ be equipped with the *product topology*. That is, we view S as $\prod_{i\in[0,1]}[0,1]_i$.

(a) Show that S is compact, hence countably compact.

(b) Show that, in spite of its compactness, S is not sequentially compact.

Solution : We are given that the space $S = [0,1]^{[0,1]}$ viewed as $\prod_{i\in[0,1]}[0,1]_i$ is equipped with the product topology.

(a) Since $[0,1]$ is compact and given the fact that any product space of compact sets is compact (by Tychonoff theorem), then $S = \prod_{i\in[0,1]}[0,1]_i$ is compact. Since any compact set is countably compact, then S is also countably compact.

(b) We are given that $S = [0,1]^{[0,1]}$ is equipped with the product topology. That is, we view S as $\prod_{j\in[0,1]}[0,1]_j$, a compact set. We will construct a sequence in S which has no converging subsequence.

Suppose each element, x, of $[0,1]$ is expressed in its binary expansion form, $[0,1]_2$. For $n \in \mathbb{N}\backslash\{0\}$ we will define the function $f_n : [0,1]_2 \to \{0,1\}$ as,

$$f_n(x) = \text{ “the } n^{th} \text{ digit in the binary expansion of } x\text{”}$$

to form a sequence $\{f_1, f_2, f_3, f_4, \ldots\}$ each mapping $[0,1]$ into $\{0,1\}$.
For example, given a particular value of $x = 0.1011101010\ldots \in [0,1]_2$

$$f_1(x) = 1,\ f_2(x) = 0,\ f_3(x) = 1,\ f_4(x) = 1\ldots$$

with which we form the ordered sequence, $\{f_1(x), f_2(x), f_3(x), f_4(x), \ldots\} = \{1,0,1,1,\ldots\}$.

Then

$$T \;=\; \{f_n : n = 1,2,3,\ldots\}$$

is a sequence of functions each mapping $[0,1]$ into $\{0,1\}$. So $T \subset \prod_{i\in[0,1]_2}\{0,1\}$.

We will show that T cannot have a convergent subsequence.

For suppose $\{f_{n_k} : k = 1,2,3,\ldots\}$ is a subsequence of T which converges to the function $f \in \prod_{i\in[0,1]_2}\{0,1\}$. Then, for every $x \in [0,1]_2$, $\{f_{n_k}(x) : k = 1,2,3,\ldots\}$ must converge to $f(x)$.

We will choose $q \in [0,1]_2$ so that $f_{n_{2k}}(q) = 0$ and $f_{n_{2k-1}}(q) = 1$.

But the subsequence,

$$\{f_{n_1}(q), f_{n_2}(q), f_{n_3}(q), \ldots, \} = \{1, 0, 1, 0, 1, \ldots\}$$

clearly, does not converge (when it should converge to $f(q)$).

So the sequence of functions T in the compact space $\prod_{i\in[0,1]_2}\{0,1\}$ does not have a convergent subsequence.

So S cannot be sequentially compact.

In section:

18.3 Some basic properties of local compactness.

Theorem 18.3 part (d)

(d) Let S be any Hausdorff topological space which contains a non-empty subspace, W. If W is locally compact, then W is the intersection of an open subset with a closed subset of S.

Proof : We are given that S is a topological space.

(d) Suppose W is a subset of a Hausdorff topological space. Suppose W is locally compact. We are required to show that W is the intersection of an open set and a closed set in S. If we can show that W is open in $\text{cl}_S W$, we are done (for, in this case, there is an open W^* in S such that $W = W^* \cap \text{cl}_S W$).

Claim: That W is open in $\text{cl}_S W$.

Proof of claim: Let $x \in W$. Given that W is locally compact and Hausdorff, we can find an open neighborhood, A, of x in W such that $\text{cl}_W A$ is compact. There is an open subset, A^*, in S such that $A = W \cap A^*$. Then

$$\text{cl}_W A = \text{cl}_S A \cap W$$

a closed subset of S (since $\text{cl}_W A$ is compact).
So,

$$\begin{aligned} x \in A \subseteq \text{cl}_S A \cap W \quad &\Rightarrow \quad \text{cl}_S A \subseteq \text{cl}_S A \cap W \\ &\Rightarrow \quad \text{cl}_S A \subseteq W \\ &\Rightarrow \quad \text{cl}_S W \cap A^* \subseteq W \end{aligned}$$

So $x \in \mathrm{cl}_S W \cap A^*$, an open neighborhood in x in $\mathrm{cl}_S W$. We have thus shown that A is open in $\mathrm{cl}_S W$, as claimed.

It then follows that $W = A^* \cap \mathrm{cl}_S W$, the intersection of an open set in S with a closed set in S.

In Chapter 19 : Exercise # 2:

2. Suppose the space S is a Lindelöf space and $\mathscr{U}$ is a **locally finite** family of subsets of S. Show that $\mathscr{U}$ is a countable family of subsets.

In Chapter 20 : Exercise # 8:

8. Suppose the space (S, τ) is equipped with the cofinite topology. Let the function $f : [0,1] \to S$ be continuous. Show that $f[[0,1]]$ **need not** be a singleton.

In section:

21.7 Pseudocompact spaces revisited.

In proof of theorem ($\Leftrightarrow$)

Suppose S is not pseudocompact. Then $C(S)$ contains an unbounded function g. Let

$$f = |g| \vee k \qquad \text{(Not } f = |g| \wedge k\text{)}$$

where $k > 0$. Then f is continuous real-valued unbounded on S. Then, for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $f(x_n) \in (n, \infty)$.

21.11 Topic: Compactifying a subset T of $S \subseteq \beta S$.

If T is a non-compact subset of S, it is interesting to reflect on how $\mathrm{cl}_{\beta S} T$ compares with βT. Does it make sense to say that $\beta T \subseteq \beta S$? Under what conditions are $\mathrm{cl}_{\beta S} T$ and βT equivalent compactifications of T? We examine this question in the following example.

Example 9. Let T be a non-empty C^*-embedded non-compact subspace of a completely regular space, S. Show that $\mathrm{cl}_{\beta S} T$ is equivalent to βT.

Solution: We are given that $T \subseteq S$ where T is non-compact and C^*-embedded in S. Since subspaces of completely regular spaces are completely regular, then T is completely regular. Since T is C^*-embedded in S and S is C^*-embedded in βS then T is C^*-embedded in βS. See that $\mathrm{cl}_{\beta S} T$ is a compactification of T. Since T is C^*-embedded in βS it is C^*-embedded in $\mathrm{cl}_{\beta S} T$. So $\mathrm{cl}_{\beta S} T$ is a compact subset of βS which is equivalent to βT.

The converse is easily verified to hold true. We state this formally in the following theorem.

Theorem 21.22 Let T be a non-empty non-compact subspace of the completely regular space S. Then T is C^*-embedded in S if and only if $\mathrm{cl}_{\beta S} T = \beta T$.

Proof: The direction ($\Rightarrow$) is proven in the example above.
($\Leftarrow$) Suppose T is such that $\mathrm{cl}_{\beta S} T = \beta T$. We are required to show that T is C^*-embedded in S. Since $\mathrm{cl}_{\beta S} T$ is a compactification equivalent to βT, T is C^*-embedded in $\mathrm{cl}_{\beta S} T$. Since $\mathrm{cl}_{\beta S} T$ is a compact subset of βS it is C^*-embedded in βS. So T is C^*-embedded in S.

In section:

22.5 More on equivalent singular compactifications.

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Add Examples 9 and 10.

In the following example we see how we can use a singular function to construct, from a rectangle, a cylindrical shell.

Example 9. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology. Consider the function $f : S \to [0, 2\pi]$ defined as $f[\{x\} \times (0, 2\pi)] = \{x\}$. Verify that the function f is continuous and that the singular set, $S(f)$, of f is the closed interval, $[0, 2\pi]$. Then verify that $f[S] \subseteq [0, 2\pi]$ so that $S \cup_f S(f)$ is a singular compactification of S. Also verify that the singular compactification of S, induced by f, is (topologically speaking) a closed and bounded cylindrical shell with radius 1.

Solution : For $x \in [0, 2\pi]$, $f^{\leftarrow}[B_\varepsilon(x)] = B_\varepsilon(x) \times [0, 2\pi]$ is open in S and so f is continuous on S. See that, for all $x \in [0, 2\pi]$, $\text{cl}_S f^{\leftarrow}[B_\varepsilon(x)]$ is not compact in S so $S(f) = [0, 2\pi]$. Since $f[S]$ is a (proper) subset of $S(f) = [0, 2\pi]$ then f is a singular map and so $S \cup_f [0, 2\pi]$ is a singular compactification of S.

Then what geometric representation can we provide for $S \cup_f [0, 2\pi]$? Well, let's consider the point x in $S(f)$ viewed as an element of the compactification $S \cup_f S(f)$. An open neighborhood base of x in $S \cup_f S(f)$ would look something like this

$$\{ \ B_\varepsilon(x) \ \cup \ [\ f^{\leftarrow}[B_\varepsilon(x)] \setminus [0, 2\pi] \times [\delta, 2\pi - \delta] \ ] : \varepsilon, \delta > 0 \ \}$$

where $[0, 2\pi] \times [\delta, 2\pi - \delta]$ is seen to be a compact subset of S. So $S(f)$ appears to be the edge which provides the material necessary to seal together the bottom and the top edges of the rectangle $[0, 2\pi] \times (0, 2\pi)$ to form a cylindrical shell.

Example 10. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology. Consider the function $g : S \to [0, 2\pi]$ defined as

$$g[\{x\} \times (0, 2\pi)] = \{2\pi - x\}$$

Verify that that $S \cup_g S(g)$ (where $S(g) = [0, 2\pi]$) is a singular compactification.
Also, if $f : S \to [0, 2\pi]$ is the function as defined in the previous example, show that $S \cup_f S(f)$ and $S \cup_g S(g)$ are not equivalent compactifications (in spite of the fact that $S(f) = S(g) = [0, 2\pi]$).

Solution : The proof that $S \cup_g S(g)$ is a singular compactification of S mimics the proof which appears in the previous example for $S \cup_f S(f)$.

Suppose $\gamma S = S \cup_f S(f)$ and $\alpha S = S \cup_g S(g)$. We verify that γS and αS are not equivalent compactifications.

Suppose $\gamma S \equiv \alpha S$. Then there exists a homeomorphism $\pi_{\gamma \to \alpha} : \gamma S \to \alpha S$ which fixes the points of S. Then $\pi_{\gamma \to \alpha}|_{\gamma S \setminus S}$ maps $[0, 2\pi]$ homeomorphically onto $[0, 2\pi]$ and so is monotone. Suppose without loss of generality that it is increasing and so maps 0 to

0 and 2π to 2π.

Consider the open ball $B = B_\varepsilon(2\pi)$ in $[0, 2\pi]$. Let

$$D = \pi_{\gamma\to\alpha}[B]$$

an open subset of $S(g)$. Then

$$\begin{aligned} f^{\leftarrow}[B] &= (2\pi - \varepsilon, 2\pi] \times (0, 2\pi) \\ g^{\leftarrow}[D] &= [0, \delta) \times (0, 2\pi) \end{aligned}$$

We can of course choose ε so that $f^{\leftarrow}[B] \cap g^{\leftarrow}[D] = \varnothing$.

Recall that $g : S \to S(g)$ extends to $g^\alpha : \alpha S \to S(g)$ such that g^α fixes the points of $S(g)$, so we have

$$\begin{aligned} g^{\alpha\leftarrow}[D] &= D \cup g^{\leftarrow}[D] \quad \text{(An open subset of } \alpha S\text{)} \\ \pi_{\gamma\to\alpha}[B \cup f^{\leftarrow}[B]] &= D \cup f^{\leftarrow}[B] \quad \text{(An open subset of } \alpha S\text{)} \end{aligned}$$

Then

$$\begin{aligned} (D \cup g^{\leftarrow}[D]) \cap (D \cup f^{\leftarrow}[B]) &= D \cup (g^{\leftarrow}[D]) \cap f^{\leftarrow}[B]) \\ &= D \cup \varnothing \\ &= D \end{aligned}$$

so D is an open subset of αS which is contained in $\alpha S \setminus S$. A contradiction! So $\gamma S \not\equiv \alpha S$. [5]

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[5]See that the compactification $S \cup_g S(g)$ constructed in this way is a Möbius strip.

Theorem 23.5 If the metrizable space S is pseudocompact then it is compact.

Proof: Suppose S is a topological space induced by a metric ρ. Also, suppose S is known to be pseudocompact. We are required to show that S is compact. If S is finite it is compact, so we need only consider the case where S is infinite. Let $A = \{x_i : i \in \mathbb{N}\backslash\{0\}\}$ be an infinite sequence in S. To show that S is compact, it suffices to show that it is sequentially compact.

We can, inductively, construct basic open neighborhoods $\{B'_{\delta_i}(x_i) : i \in \mathbb{N}\backslash\{0\}\}$ such, for each i,

$$B'_{\delta_i}(x_i) \ \cap \ \cup\{\mathrm{cl}_S B'_{\delta_j}(x_j) : j < i\} = \varnothing$$

To make the open balls nested, for each i, let

$$B_{\delta_i}(x_i) = B'_{\delta_i}(x_i) \cap B'_{1/i}(x_i)$$

For each j, let $U_j = \cup\{B_{\delta_i}(x_i) : i > j\}$ and $F_j = \mathrm{cl}_S[U_j]$. Then both $\{U_j : j \in \mathbb{N}\backslash\{0\}\}$ and $\{F_j : j \in \mathbb{N}\backslash\{0\}\}$ are nested collections of sets where

$$F_j \ \cap \ \cup\{B_{\delta_i}(x_i) : i < j\} = \varnothing$$

Since S is completely regular and pseudocompact, then S is feebly compact. So by the characterization of feebly compact

$$\cap\{F_j : j \in \mathbb{N}\backslash\{0\}\} \neq \varnothing$$

Suppose, now, that $t \in \cap\{F_j\}$ and let $B_\varepsilon(t)$ be a basic open neighborhood of t. Since t belongs to each $F_j = \mathrm{cl}_S[U_j]$, then $B_\varepsilon(t)$ intersects each $\mathrm{cl}_S[U_j]$, and so intersects each $U_j = \cup\{B_{\delta_i}(x_i) : i > j\}$. Then for each j, $B_{1/j(i)}(t)$ intersects $B_{\delta_{j(i)}}(x_{j(i)})$ for some $j(i) > j$. Say, $a_{j(i)}$ belongs to this intersection. Then

$$\rho(t, x_{j(i)}) \ \leq \ \rho(t, a_{j(i)}) + \rho(a_{j(i)}, x_{j(i)}) \ < \ 1/j(i) + 1/j(i)$$

Then $\lim_{i\to\infty} \rho(t, x_{j(i)}) = 0$. Then t is the limit of a subsequence, $\{x_{j(i)} : i \in \mathbb{N}\backslash\{0\}\}$, of $A = \{x_i : i \in \mathbb{N}\backslash\{0\}\}$. So every sequence of the metric space, S, has a convergent sequence, which testifies to the fact that S is compact. Then, if the metric space S is pseudocompact, it is compact.

In section:

24.1 Realcompact space: Definitions and characterizations.

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Definition 24.1: Let S be a completely regular space.

a) If $f \in C(S)$ ...

All through subsection 24.1 theorem statements, replace the expression

"locally compact and Hausdorff"
with
"completely regular".

In section:

24.3 The *Hewitt-Nachbin realcompactification* of a space S.

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Add:

Theorem 24.7 Suppose S is completely regular. The space S is realcompact if and only if S is homeomorphic to a closed subspace of a power of $\mathbb{R}$.

Proof: We are given that S is completely regular. Then S is densely contained into its realcompactification υS. Then every (real-valued) function f in $C(S)$ extends continuously to (a real-valued function) $f^\upsilon \in C(\upsilon S)$. Let $\mathscr{F} = C(S)$. Then the evaluation map $e_\mathscr{F} : S \to \prod_{f\in\mathscr{F}} \mathbb{R}_f$ extends continuously to the evaluation map on υS defined as

$$e^\upsilon_\mathscr{F}(x) =< f^\upsilon(x) >_{f\in\mathscr{F}} \in \prod_{f\in\mathscr{F}} \mathbb{R}_f$$

Since υS is completely regular (given that $\upsilon S \subseteq \beta S$) then $C(\upsilon S)$ separates points and closed sets of υS. Then $e^\upsilon_\mathscr{F}$ maps υS homeomorphically onto a subset of $\prod_{f\in\mathscr{F}} \mathbb{R}_f$

(as argued in the embedding theorem I in 7.17).

Claim: We claim that $e^{\upsilon}_{\mathscr{F}}[\upsilon S]$ is a closed subset of $\prod_{f\in\mathscr{F}}\mathbb{R}_f$.

Proof of claim: Recall that the evaluation map,

$$e^{f(\omega)}_{C(\beta S,\,\omega\mathbb{R})} : \beta S \to \prod_{f\in C(\beta S,\omega\mathbb{R})} \omega\mathbb{R}_f$$

maps βS continuously onto a *compact* subset, say K, of $\prod_{f\in C(\beta S,\omega\mathbb{R})}\omega\mathbb{R}_f$ with the function defined as

$$e^{\beta(\omega)}_{C(\beta S,\,\omega\mathbb{R})}(x) \ = < f^{\beta(\omega)}(x) >_{f\in C(\beta S,\omega\mathbb{R})} \in K$$

Let $\mathscr{G} = \{\pi_f : f \in \mathscr{F}\}$ where $\pi_f : K \to \mathbb{R}_f$ is defined as

$$\pi_g[< f^{\upsilon}(x) >_{f\in\mathscr{F}}] = g^{\upsilon}(x) \in \mathbb{R}_g$$

See that $\mathscr{G}$ separates points and closed sets of K (check!) and that the function $g : K \to \prod_{f\in\mathbb{R}_f}$ defined as,

$$g(x) = e_{\mathscr{G}}(x) \ = < \pi_f(x) >_{\pi_f\in\mathscr{G}} = < f^{\upsilon}(x) >_{f\in\mathscr{F}} = \ e^{\upsilon}_{\mathscr{F}}(x) \in \prod_{f\in\mathscr{F}} \mathbb{R}_f$$

so $g\circ e^{\beta(\omega)}_{C(\beta S,\,\omega\mathbb{R})}[\beta S] = g[K] = e^{\upsilon}_{\mathscr{F}}[\upsilon S]$ is closed in $\prod_{f\in\mathscr{F}}\mathbb{R}_f$, as claimed.

Let $A = \prod_{f\in\mathscr{F}}\mathbb{R}$. We then have

$$\begin{aligned} e_{\mathscr{F}}[S] &\subseteq \mathrm{cl}_A e_{\mathscr{F}}[S] \\ &\subseteq e^{\upsilon}_{\mathscr{F}}[\upsilon S] \quad \text{(Since } e^{\upsilon}_{\mathscr{F}}[\upsilon S] \text{ is closed in } A\text{)} \\ &= e^{\upsilon}_{\mathscr{F}}[\mathrm{cl}_{\upsilon S} S] \\ &\subseteq \mathrm{cl}_A e_{\mathscr{F}}[S] \quad \text{(By continuity of } e^{\upsilon}_{\mathscr{F}}.\text{)} \end{aligned}$$

So

$$e^{\upsilon}_{\mathscr{F}}[\upsilon S] = \mathrm{cl}_A e_{\mathscr{F}}[S]$$

We conclude,

$$\begin{aligned} S = \upsilon S \ &\Leftrightarrow \ e^{\upsilon}_{\mathscr{F}}[\upsilon S] = \mathrm{cl}_A e_{\mathscr{F}}[S] = e_{\mathscr{F}}[S] \\ &\Leftrightarrow \ e_{\mathscr{F}}[S] = \mathrm{cl}_A e_{\mathscr{F}}[S] \subseteq A \end{aligned}$$

We conclude that S is realcompact if and only if the homeomorphism, $e_{\mathscr{F}}$, maps S onto a closed subset of a power of $\mathbb{R}$.

In section:

29.2 A base for a uniformity.

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Example 3. Consider the set of real numbers $\mathbb{R}$. For $\kappa > 0$, let

$$B_\kappa = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - b| < \kappa\}$$

Verify that the collection,

$$\mathscr{B} = \{B_\kappa : \kappa > 0\}$$

forms a base for some uniformity on $\mathbb{R}$.

Solution : We verify that $\mathscr{B}$ satisfies the four base properties for a uniformity.

U1. Since $(x, x) \in B_\kappa$ for all κ, then $\mathscr{B}$ satisfies U1.

U2*. For κ_1 and κ_2 larger than 0, let $\kappa_3 = \min\{\kappa_1, \kappa_2\}$. If $(x, y) \in B_{\kappa_3}$, then (x, y) are strictly within a distance of κ_3 from each other. So $(x, y) \in B_{\kappa_1} \cap B_{\kappa_2}$. Then $B_{\kappa_3} \subseteq B_{\kappa_1} \cap B_{\kappa_2}$. It follows that $\mathscr{B}$ satisfies U2*.

U3. Let $B_\kappa \in \mathscr{B}$. We claim there exists λ such that $B_\lambda \circ B_\lambda \subseteq B_\kappa$.

Let $\lambda = \kappa/4$. Recall that

$$U \circ V = \{(u, v) : (u, y) \in V \text{ and } (y, v) \in U \text{ for some } y \in \text{im } V.\}$$

Let $(u, v) \in B_{\kappa/4} \circ B_{\kappa/4}$. We claim that $|u - v| < \kappa$.

See that $(u, v) \in B_{\kappa/4} \circ B_{\kappa/4}$ implies that there exists $z \in \text{ im } B_{\kappa/4}$ such that $(u, z) \in B_{\kappa/4}$ and $(z, v) \in B_{\kappa/4}$.

Then $|u - z| < \kappa/4$ and $|z - v| \in \kappa/4$. We have,

$$\begin{aligned} |u - v| &\leq |u - z| + |z - v| \\ &< \kappa/4 + \kappa/4 \\ &= \kappa/2 < \kappa \end{aligned}$$

Then $(u, v) \in B_\kappa$. So $B_{\kappa/4} \circ B_{\kappa/4} \subseteq B_\kappa$.

We conclude that $\mathscr{B}$ satisfies U3.

U4. Let $B_\kappa \in \mathscr{B}$. Since $B_\kappa = \{(x,y) : |x-y| < \kappa\} = \{(x,y) : |y-x| < \kappa\} = B_\kappa^{-1}$, then $\mathscr{B}$ satisfies U4.

Then

$$\mathscr{B} = \{B_\kappa : \kappa > 0\}$$

is a base which generates a uniformity on $\mathbb{R}$. Then the uniformity on $\mathbb{R}$ is

$$\mathscr{U} = \{V \in \mathscr{P}(\mathbb{R} \times \mathbb{R}) : B_\kappa \subseteq V \text{ for some } \kappa > 0\}$$

In section:

29.4 The uniform topology, $\tau_{\mathscr{U}}$, generated by a uniformity, $\mathscr{U}$, on a set.

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Theorem 29.8 Let S be a non-empty set, $\mathscr{U}$ be a uniformity on S and $\mathscr{B}$ be a base for $\mathscr{U}$. For $x \in S$, and B in $\mathscr{U}$, let $B(x)$ denote the image of $\{x\}$ (in S) under the relation, B. Let

$$\tau_{\mathscr{U}} = \{U \in \mathscr{P}(S) : \text{ for each } x \in U \text{ there exists } B \in \mathscr{U} \text{ such that } B(x) \subseteq U\}$$

Then $\tau_{\mathscr{U}}$ is a topology on S.

Add Theorem 29.10:

Theorem 29.10: Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. Let $\tau_{\mathscr{U}}$ be the topology on S generated by the uniformity, $\mathscr{U}$. If $B \in \mathscr{U}$ and $x \in S$, let $B(x)$ denote the image of $\{x\}$ under B. Let T be a subset of S and $\text{int}_S T$ be the non-empty interior of T with respect to the uniform topology, $\tau_{\mathscr{U}}$, on S. Then $x \in \text{int}_S T$ if and only if there is some $B \in \mathscr{U}$ such that $B(x) \subseteq T$. Hence

$$\mathscr{B}(x) = \{B(x) : B \in \mathscr{U}\}$$

forms a base for the neighborhood system of x.

Proof: Let

$$M = \{x \in S : B(x) \subseteq T\}$$

We claim that M is open with respect to the uniform topology. By theorem 29.8, to show this, it suffices to verify that, for every x in M, there is a $B \in \mathscr{U}$ such that $B(x) \subseteq M$.

Proof of claim: Suppose $x \in M$. This means there is some $B \in \mathscr{U}$ such that $x \in B(x) \subseteq T$. By U3, there exists $V \in \mathscr{U}$ such that $V \circ V \subseteq B$. Suppose $z \in V(x)$. Then

$$V(z) \subseteq V \circ V(x) \subseteq B(x) \subseteq T$$

Since $V(z) \subseteq T$ then $z \in M$. So every element of $V(x)$ belongs to M. We conclude that $V(x) \subseteq M$. We have shown that for any $x \in M$ there is a $V \in \mathscr{U}$ such that $V(x) \subseteq T$. By theorem 29.8, M is open with respect to the uniform topology. This establishes the claim.

Since M is the largest open subset of S which is contained in T then $M = \text{int}_S T$.

We can then conclude that for any neighborhood T of x in S there is a $B \in \mathscr{U}$ such that $x \in B(x) \subseteq T$. So for each $x \in S$, $\mathscr{B}(x) = \{B(x) : B \in \mathscr{U}\}$ forms a base for a neighbourhood system of x.

...

At the end of Example 6:

See that,

$$\begin{aligned} x &\in (x - \varepsilon/4, x + \varepsilon/4) \\ &= \pi_2\left[\ (\{x\} \times S\ ) \cap B_{\varepsilon/2}\right] \\ &= (B_{\varepsilon/2})(x) \\ &\subseteq (x - \varepsilon, x + \varepsilon) \\ &\subseteq A \end{aligned}$$

Then $x \in (B_{\varepsilon/2})(x) \subseteq (x - \varepsilon, x + \varepsilon) \subseteq A$.

We conclude that $\tau \subseteq \tau_{\mathscr{U}}$, as claimed.

So $\tau_{\mathscr{U}} = \tau$.

In section:

30.2 A consequence of the *Stone-Weierstrass theorem*.

Let $\mathscr{C} = \{\alpha_i S : i \in I\}$ represent the family of all compactifications of a locally compact completely regular space. (Not locally finite....)

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