Decompositions of Grothendieck Polynomials

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Dominic Searles (USC)

A geometric question

- Let $X = \mathrm{Flags}(\mathbb{C}^n) = \mathsf{GL}_n(\mathbb{C})/\mathsf{B}$ be the parameter space of complete flags

$$\mathbb{C}^0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$$

- B-orbit closures are the Schubert varieties X_w, indexed by permutations
- These yield a $\mathbb{Z}$-module basis $\{[X_w]\}$ of the cohomology $H^\star(X)$

Question

What are the structure coefficients of this algebra?

$$[X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w]$$

Since $c_{u,v}^w \in \mathbb{N}$, it **should** *be possible to express $c_{u,v}^w$ as the cardinality of some set of combinatorial objects.*

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Now, $c_{u,v}^w \in \mathbb{N}[\beta]$ [Brion, '02]. We recover $H^\star(X)$ at $\beta = 0$ and $K(X)$ at $\beta \neq 0$.

A combinatorial tool

- The β-Grothendieck polynomials [Lascoux-Schützenberger '82, Fomin-Kirillov '94] are polynomial representatives:

$$\mathfrak{G}_u \cdot \mathfrak{G}_v = \sum_w c_{uv}^w \mathfrak{G}_w$$

- $\{\beta^k \mathfrak{G}_w\}$ is an additive basis for $\mathrm{Poly}_n[\beta] := \mathbb{Z}[x_1, \ldots, x_n, \beta]$.

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Example (Silly!)

$D_u = \mathfrak{G}_u$:

1. Trivial
2. Very hard!
3. Trivial

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Example (Slightly less silly)

D-basis = monomial basis:

1. Nice rule
2. Trivial
3. Most of the difficulty got shuffled over here

New example: Glide polynomials

Example (P.-Searles '17)

We introduce the **glide polynomials** $\{\mathcal{G}_a\}$, a new basis of $\mathrm{Poly}_n[\beta]$:

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Glide polynomials are indexed by weak compositions (e.g. $a = 01003$). A colored weak composition is a **glide** of a if it can be obtained by a sequence of the following local moves:

1. $0k \rightsquigarrow k0$
2. $0k \rightsquigarrow ij$ $(i, j \geq 0,\ i + j = k)$
3. $0k \rightsquigarrow i(j+1)$ $(i, j \geq 0,\ i + j = k)$

Glide polynomials

Definition (P.-Searles '17)

The glide polynomial is

$$\mathcal{G}_a = \sum_{\text{glides } b \text{ of } a} \beta^{\#\text{red}} x_1^{b_1} \cdots x_n^{b_n}$$

Example

$$\mathcal{G}_{0102} = \mathbf{x}^{0102} + \mathbf{x}^{1002} + \mathbf{x}^{0120} + \mathbf{x}^{1020} + \mathbf{x}^{1200} + \mathbf{x}^{0111} + \mathbf{x}^{1011} + \mathbf{x}^{1101} + \mathbf{x}^{1110} + \beta\mathbf{x}^{0112} + 2\beta\mathbf{x}^{1102} + 2\beta\mathbf{x}^{1120} + \beta\mathbf{x}^{1021} + \beta\mathbf{x}^{0121} + 3\beta\mathbf{x}^{1111} + \beta\mathbf{x}^{1210} + \beta\mathbf{x}^{1201} + 2\beta^2\mathbf{x}^{1112} + 2\beta^2\mathbf{x}^{1121} + \beta^2\mathbf{x}^{1211}$$

Key properties

Theorem (P.-Searles '17)

- $\{\beta^k \mathcal{G}_a\}$ *is a basis for* $\mathrm{Poly}_n[\beta]$.
- β*-Grothendieck polynomials expand positively:*

$$\mathfrak{G}_w = \sum_a e_a^w \mathcal{G}_a,$$

where $e_a^w = \beta^{|a|-|w|} \cdot \#QY$ *pipe dreams for* w *of weight* a.

Example

$\mathfrak{G}_{12543}$ is sum of 68 monomials, but only 9 glide polynomials:

$$\mathfrak{G}_{12543} = \mathcal{G}_{0021} + \mathcal{G}_{0120} + \beta\mathcal{G}_{0121} + \beta\mathcal{G}_{0220} + \beta^2\mathcal{G}_{0221} + \beta^2\mathcal{G}_{1220} + \beta^3\mathcal{G}_{1221} + \beta^3\mathcal{G}_{2220} + \beta^4\mathcal{G}_{2221}$$

Key properties (cont.)

Theorem (P.-Searles '17)

$\{\beta^k \mathcal{G}_a\}$ *has positive structure coefficients:*

$$\mathcal{G}_a \cdot \mathcal{G}_b = \sum_c \beta^{|c|-|a|-|b|} g_{a,b}^c \mathcal{G}_c,$$

where $g_{a,b}^c$ *is the multiplicity of* c *in* $a \sqcup_{\mathrm{gen}} b$.

Conjecture (P.-Emily Sergel)

Fix a, b. *Then*

$$\sum_c (-1)^{|c|-|a|-|b|} g_{a,b}^c = 1.$$

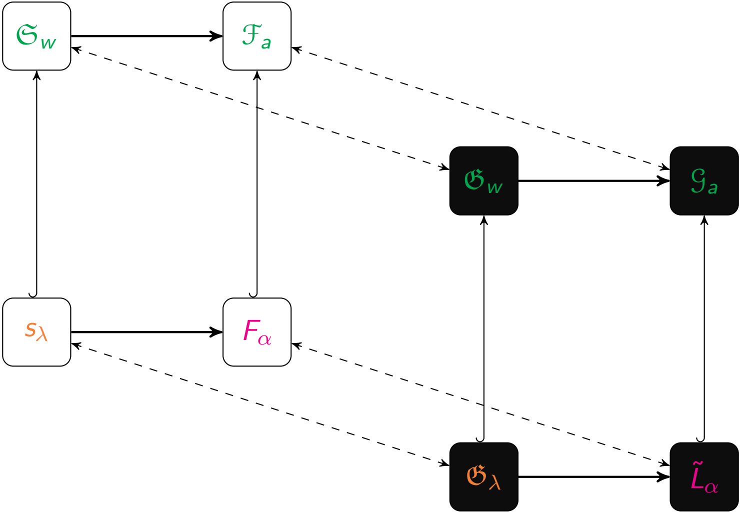
Relations to other bases

- If $\mathfrak{G}_w = \mathfrak{G}_\lambda = \sum_a \mathcal{G}_a$ is symmetric, each $\mathcal{G}_a$ is quasisymmetric.
- The quasisymmetric glides are the **multi-fundamental** basis [Lam-Pylyavskyy '07] of $\mathrm{QSym_n}[\beta]$
- Stable limits are multi-fundamental quasisymmetric functions:

$$\lim_{m \to \infty} \mathcal{G}^{(1)}_{0^m a} = \tilde{L}_{\mathrm{flat}(a)}$$

- Specializing $\beta = 0$ gives the **slide** basis [Assaf-Searles '17] of Poly_n.

Relations to other bases



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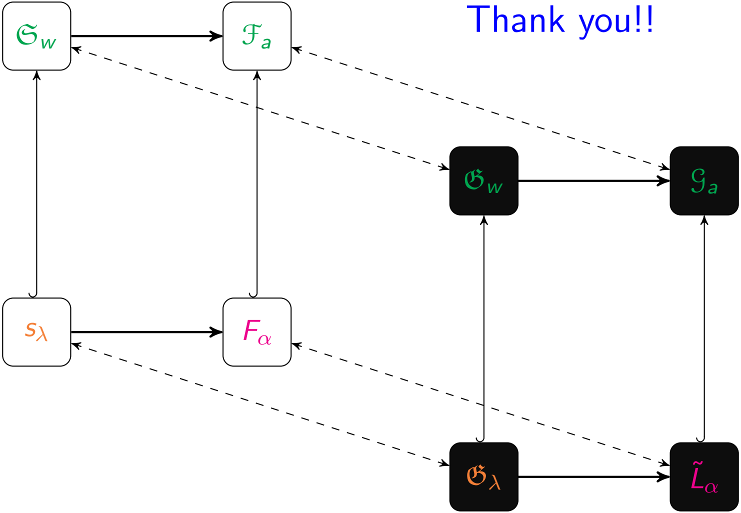
$\mathfrak{S}_w$
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s_λ
F_α
$\mathfrak{G}_\lambda$
$\tilde{L}_\alpha$