The Rise-Contact involution on Tamari intervals

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Motivations: Tamari intervals (they’re cool)

2005 Chapoton proves a nice formula counting intervals of the Tamari lattice (also counting triangular maps!)

\[
\frac{2}{n(n+1)} \binom{4n+1}{n-1}
\]

Motivations: Tamari intervals (they’re cool)

2010 Bergeron and Préville-Ratelle give the definition of $m$-Tamari lattices and conjecture that the number of intervals is counted by

$$\frac{m + 1}{n(mn + 1)} \binom{(m + 1)^2 n + m}{n - 1}$$

Motivations: Tamari intervals (they’re cool)

2011 Bousquet-Mélou, Fusy, and Préville-Ratelle prove the formula for $m$-Tamari intervals.

The generating function shows a symmetry between the number of contacts and the initial rises but they have no combinatorial explanation.

Motivations: Tamari intervals (they’re cool)

2012 Préville-Ratelle leaves a conjecture at this end of his thesis about the symmetry of two series of statistics “contacts” and “rises”

Motivations: Tamari intervals (they’re cool)

Today, I give you the involution that proves this symmetry.

The Rise-Contact involution on Tamari intervals
Contacts : 2
Contacts : 2
Contact vector: 2,
Contacts : 2
Contact vector: 2, 2,
Contacts : 2
Contact vector: 2, 2, 0,
Contacts: 2
Contact vector: 2, 2, 0, 1,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0,
Contacts: 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1, 0
The Rise-Contact involution on Tamari intervals
Rises : 2
Rises : 2
Rises vector: 2,
Rises : 2
Rises vector: 2, 2,
Rises : 2
Rises vector: 2, 2, 0,
Rises : 2
Rises vector: 2, 2, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0, 0
The Rise-Contact involution on Tamari intervals.
The Rise-Contact involution on Tamari intervals.
Contact vector: 3, 0, 2, 0, 0, 4, 0, 0, 1, 0
Rise vector: 3, 1, 0, 2, 3, 0, 1, 0, 0, 0
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Dyck paths
Tamari lattice

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The Rise-Contact involution on Tamari intervals
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

The Rise-Contact involution on Tamari intervals
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

↓ \Phi
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

\[ \Phi \]

Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

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The Rise-Contact involution on Tamari intervals
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Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

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The Rise-Contact involution on Tamari intervals
Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

↓ Ψ
Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

\[ \downarrow \psi \]

Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
The Rise-Contact involution on Tamari intervals
Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

↓ \Phi
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Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

↓ \Phi

Contact vector: 2, 2, 0, 2, 0, 2, 0, 0, 2, 0
Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

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The Rise-Contact involution on Tamari intervals
The Rise-Contact involution on Tamari intervals

Contact vector: $2, 2, 0, 1, 0, 2, 0, 2, 0, 1 - 2^4, 1^2, 0^4$
Rise vector: $2, 2, 0, 0, 2, 2, 2, 0, 0, 0 - 2^5, 0^5$

$\downarrow \Phi \circ \Psi \circ \Phi$

Contact vector: $2, 2, 0, 2, 0, 2, 0, 0, 2, 0 - 2^5, 0^5$
Rise vector: $2, 2, 2, 1, 0, 0, 0, 2, 1, 0 - 2^4, 1^2, 0^4$
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

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The Rise-Contact involution on Tamari intervals
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

\[ \Phi \]
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0
Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0
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The Rise-Contact involution on Tamari intervals

Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

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Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

↓ \( \Psi \)

Contact vector: 1, 2, 0, 3, 2, 0, 0, 0
rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0

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Contact vector: 1, 2, 0, 3, 2, 0, 0, 0
rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0

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The Rise-Contact involution on Tamari intervals
Contact vector: 1, 2, 0, 3, 2, 0, 0, 0
Rise vector: 4, 2, 0, 1, 0, 0, 1, 0

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The Rise-Contact involution on Tamari intervals
Contact vector: $4, 0, 1, 0, 2, 0, 0, 1 - 4, 2, 1^2, 0^4$
Rise vector: $1, 2, 0, 3, 2, 0, 0, 0 - 3, 2^2, 1, 0^4$

\[ \Phi \circ \Psi \circ \Phi \]

Contact vector: $1, 2, 3, 2, 0, 0, 0, 0 - 3, 2^2, 1, 0^4$
Rise vector: $4, 2, 0, 1, 0, 0, 1, 0 - 4, 2, 1^2, 0^4$
A combinatorial bijection which switches the contacts and the rises

Can be extended to the $m$-Tamari case

Preserve the “distance” of the interval


**SageMath Demo** :
github.com/VivianePons/public-notebooks/
“distance” preserving
$m$-Tamari

$m$-Contact vector: 5, 0, 0, 1, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0

$m$-Rise vector: 1, 0, 2, 0, 0, 1, 0, 2, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0
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