CM regularity and Kazhdan-Lusztig varieties

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Introduction

We give an explicit formula for the degree of the Grothendieck polynomial of a Grassmannian permutation w and a closely related formula for the CM regularity of the Schubert determinantal ideal of w. We then resolve a conjecture of Kummini-Lakshmibai-Sastry-Seshadri.

Background

Consider the Schubert variety X_v and the opposite Schubert cell Ω_w° in $\mathbf{GL}_n(\mathbb{C})$.

Theorem 1 (Kazhdan-Lusztig, 1979)

$$X_v \cap w\Omega_{id}^\circ \cong (X_v \cap \Omega_w^\circ) \times \mathbb{A}^{\ell(w)}$$

Of particular interest is the **Kazhdan-Lusztig variety**

$$\mathcal{N}_{v,w} = X_v \cap \Omega_w^\circ,$$

with coordinate ring $R_{v,w}$. For $w \leq v$, $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathrm{z}_{i\times j}(v)\rangle.$$

Example 2 *Consider* $w = 4132, v = 4231.$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathrm{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31}\rangle$$

Consider the coordinate ring S/I, where S is standard graded. We consider its **minimal free resolution**

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}(S/I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(S/I)} \to S/I \to 0$$

The **Castelnuovo-Mumford regularity** of S/I

$$\mathrm{reg}(S/I) := \max\{j - i \mid \beta_{i,j}(S/I) \neq 0\}.$$

Proposition 3 *For Cohen-Macaulay* S/I

$$\textit{reg}(S/I) = \textit{deg}\ \mathcal{K}(S/I;\mathrm{t}) - \textit{ht}_S I$$

where $\mathcal{K}(S/I;\mathrm{t})$ *is the* K*-polynomial of* S/I.

Matrix Schubert varieties $\overline{X}_{w'}$ are special cases of $\mathcal{N}_{v,w}$. Combining results of Fulton and Knutson-Miller,

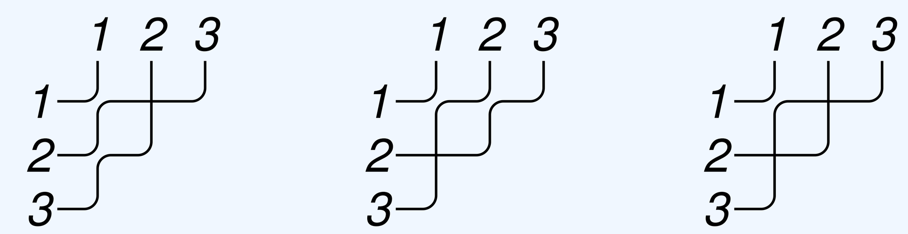
Theorem 4 *For* $w \in \mathcal{S}_n$, *reg*$(R_w) =$ *deg* $(\mathfrak{G}_w) - \ell(w)$, *where* R_w *denotes the coordinate ring of* $\overline{X}_w$.

The **Grothendieck polynomials** $\mathfrak{G}_w$ are the polynomial representatives of K-theoretic Schubert classes. To compute $\mathfrak{G}_w$,

$$\mathfrak{G}_w(x_1,\ldots,x_n) = \sum_{P\in\mathbf{PD}(w)} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}.$$

Example 5 *Consider* $w = 132.$

$\mathbf{PD}(w)$:

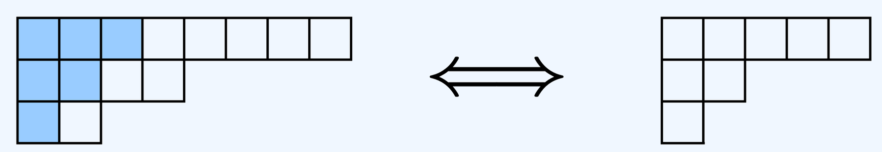


so $\mathfrak{G}_w(x_1,x_2,x_3) = x_1 + x_2 - x_1x_2.$

Grassmannian Permutations

To $w \in \mathcal{S}_n$ **Grassmannian** with descent k, we can uniquely associate a partition λ with k parts.

Example 6 *To* $w_\lambda = 24813567$ *we associate* $\lambda = (5,2,1).$



Let $B_i \subset \lambda$ denote the boxes of λ strictly northeast of (i, λ_i) and

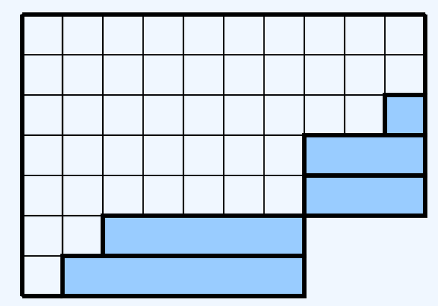
$$\mathbf{sv}(\lambda) = \max\left\{k \mid \delta^k \subseteq \lambda\right\}.$$

For λ as in Example 6, $B_3 = (4,1)$ and $\mathbf{sv}(\lambda) = 2$. Using a theorem of Lenart, we obtain the following result.

Theorem 7 (Rajchgot-Ren-R-St.Dizier-Weigandt, 2019) *Suppose* $w_\lambda \in \mathcal{S}_n$ *has descent* k. *Then*

$$\mathbf{deg}(\mathfrak{G}_{w_\lambda}) = |\lambda| + \sum_{i\in[k]} \mathbf{sv}(B_i).$$

Example 8 *Consider* $\lambda = (10,10,9,7,7,2,1)$. *Then*



$$\mathbf{deg}(\mathfrak{G}_{w_\lambda}) = 46 + (1+3+3+5+6) = 46 + 18 = 64.$$

With the fact that $\ell(w_\lambda) = |\lambda|$, we have the following corollary.

Corollary 9 (Rajchgot-Ren-R-St.Dizier-Weigandt, 2019) *Suppose* $w_\lambda \in \mathcal{S}_n$ *has descent* k. *Then*

$$\textit{reg}(R_{w_\lambda}) = \sum_{i\in[k]} \mathbf{sv}(B_i).$$

For λ as in Example 8, we have reg$(R_{w_\lambda}) = 18$.

Application

Conjecture 10 (Kummini et al., 2015) *For certain* $w_\lambda \in \mathcal{S}_n$ *with descent* k, $v = [k, k+1, \ldots, n, 1, 2, \ldots, k-1] \in \mathcal{S}_n$,

$$\textit{reg}(R_{v,w_\lambda}) = \sum_{i\in[k-1]} i \cdot (w_\lambda(k-i+1) - w_\lambda(k-i) - 1).$$

Since those $w_\lambda \in \mathcal{S}_n$ considered by Kummini et al. satisfy $\mathcal{E}ss(w_\lambda) \subset k \times (n-k)$, we have the following result:

Corollary 11 (Rajchgot-Ren-R-St.Dizier-Weigandt, 2019) *Suppose* $w_\lambda \in \mathcal{S}_n$ *has descent* k. *Then*

$$\textit{reg}(R_{v,w_\lambda}) = \sum_{i\in[k]} \mathbf{sv}(B_i).$$

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