Complete quadrics and algebraic statistics

Tim Seynnaeve

Max Planck Institute for Mathematics in Sciences, Leipzig

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Want to know more?
Check out my NASO talk, available at
https://www.mis.mpg.de/nlalg/seminars/naso.html.

Three equivalent definitions of the ML-degree $\phi(n,d)$

- $\phi(n,d)$ is the *maximum likelihood degree* of the *linear concentration model* defined by a generic d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{R}^n$.
- $\phi(n,d)$ is the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{C}^n$.
- $\phi(n,d)$ is the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2}-d$ given points and are tangent to $(d-1)$ given hyperplanes.

Complete quadrics

- The space Φ of complete quadrics is the closure of the image of

$$\begin{aligned}\mathbb{P}(S^2\mathbb{C}^n) &\dashrightarrow \mathbb{P}\left(S^2\mathbb{C}^n\right)\times\mathbb{P}\left(S^2(\textstyle\bigwedge^2\mathbb{C}^n)\right)\times\cdots\times\mathbb{P}\left(S^2(\textstyle\bigwedge^{n-1}\mathbb{C}^n)\right)\\ M &\mapsto (M,\wedge^2 M,\ldots,\wedge^{n-1}M)\end{aligned}$$

- Then $\phi(n,d)$ is the degree of the product $\mu_1^{d-1}\mu_{n-1}^{\binom{n+1}{2}-d}$ in the Chow ring (cohomology ring) $A(\Phi)$, where $\mu_i\in A^1(\Phi)$ is the pullback of the hyperplane class in $\mathbb{P}(S^2(\wedge^i\mathbb{C}^n))$.

$$\psi_\lambda=\sum_\nu\begin{vmatrix}\binom{\lambda_1+k-1}{\nu_1+k-1} & \binom{\lambda_1+k-1}{\nu_2+k-2} & \cdots & \binom{\lambda_1+k-1}{\nu_k}\\ \binom{\lambda_2+k-2}{\nu_1+k-1} & \binom{\lambda_2+k-2}{\nu_2+k-2} & \cdots & \binom{\lambda_2+k-2}{\nu_k}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{\lambda_k}{\nu_1+k-1} & \binom{\lambda_k}{\nu_2+k-2} & \cdots & \binom{\lambda_k}{\nu_k}\end{vmatrix}$$

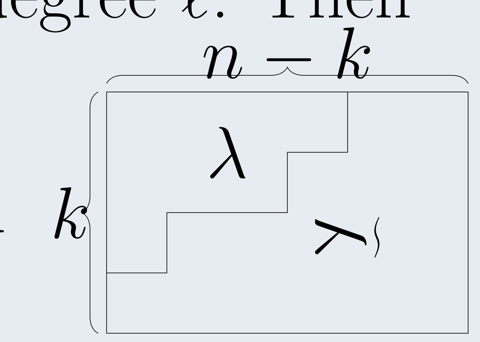
A formula for $\phi(n,d)$

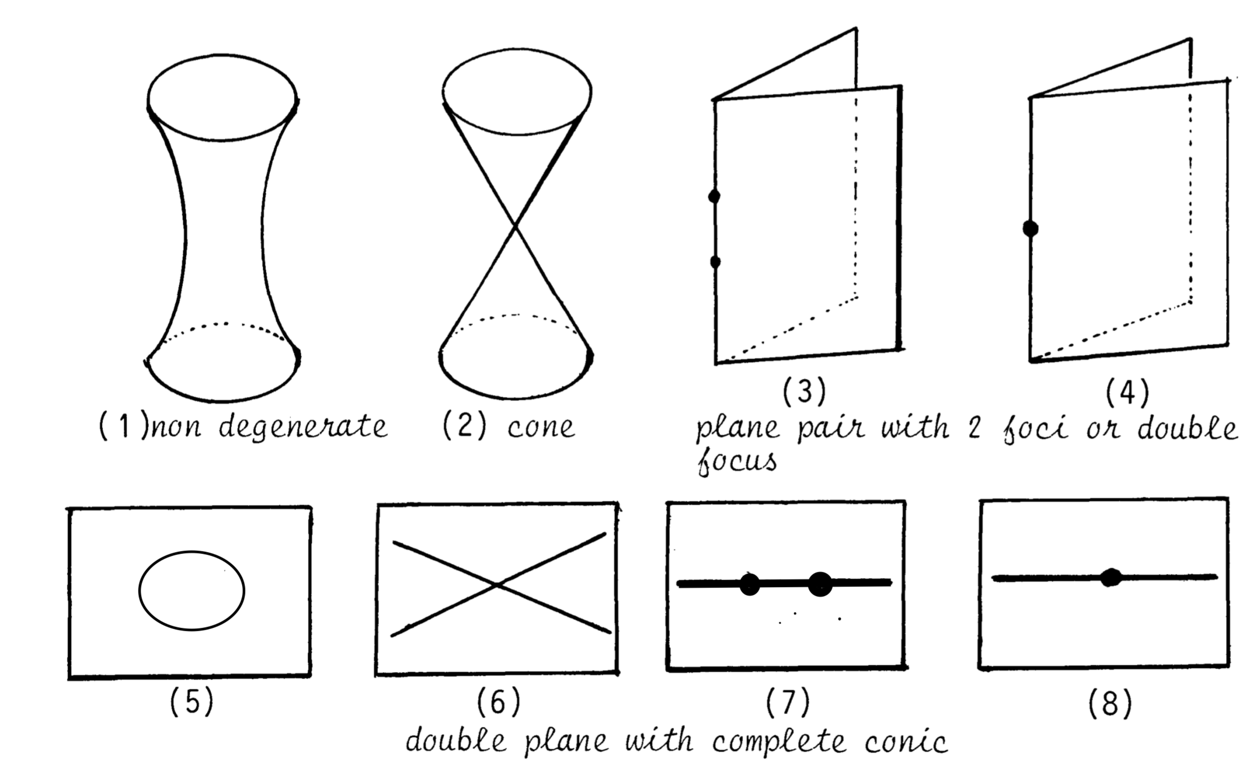
Define ψ_λ as the coefficients in the Schur decomposition of

$$H_\ell(x_i+x_j\mid 1\le i\le j\le k)=\sum_{\lambda\vdash\ell}\psi_\lambda S_\lambda(x_1,\ldots,x_k),$$

where H_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. Then

$$\boxed{\phi(n,d)=\frac{1}{n}\sum_{k=1}^{n}k\left(\sum_\lambda\psi_\lambda\psi_{\tilde\lambda}\right)},\quad\text{where } \lambda\vdash d-\binom{k+1}{2},\quad\text{and}$$





Idea of the proof

- *Degeneration class* $\delta_k:=[S_k]\in A^1(\Phi)$, where $S_k:=\{(M_1,\ldots,M_{n-1})\in\Phi\mid \mathrm{rk}(M_k)=1\}$.
- Using $2\mu_k=\mu_{k-1}+\delta_k+\mu_{k+1}$: suffices to compute $\mu_1^a\mu_{n-1}^b\delta_k$ for $a+b=\binom{n+1}{2}-2$.
- Pushforward along $\pi:S_k\to Gr(k,V)$: suffices to compute $\pi_*(\mu_1^a\mu_{n-1}^b)\in A(Gr(k,n))$.
- The Chow ring $A(Gr(k,n))$ is a quotient of the ring of symmetric functions.

Polynomiality and computations

Using our formula, and recursive relations between the ψ_λ, we can prove the following:

For fixed d, $\phi(n,d)$ is a polynomial in n, of degree $d-1$.

Moreover, we have an algorithm for computing these polynomials.
While previously, only the cases $d\le 5$ were known, our algorithm can compute $\phi(n,d)$ for $d\le 47$ in $\lesssim 5$ minutes. For instance:

$$\begin{aligned}\phi(n,18)=\frac{1}{355687428096000}&(n-5)(n-4)(n-3)(n-2)(n-1)(3024902557n^{12}-111489409997n^{11}+1862235028288n^{10}-18676382506290n^9+125446336704681n^8-594987544526781n^7+\\ &2047718727437714n^6-5214795516381220n^5+10138037306327912n^4-15696938913831072n^3+18622763914779648n^2-12286614789872640n+2964061900800).\end{aligned}$$

Based on joint work in progress with Laurent Manivel, Mateusz Michałek, Leonid Monin, Martin Vodicka, Andrzej Weber, and Jarosław, Wiśniweski

Complete quadrics and algebraic statistics

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Tim Seynnaeve

Max Planck Institute for Mathematics in Sciences, Leipzig

Want to know more?
Check out my NASO talk, available at https://www.mis.mpg.de/nlalg/seminars/naso.html.

Three equivalent definitions of the ML-degree $\phi(n,d)$

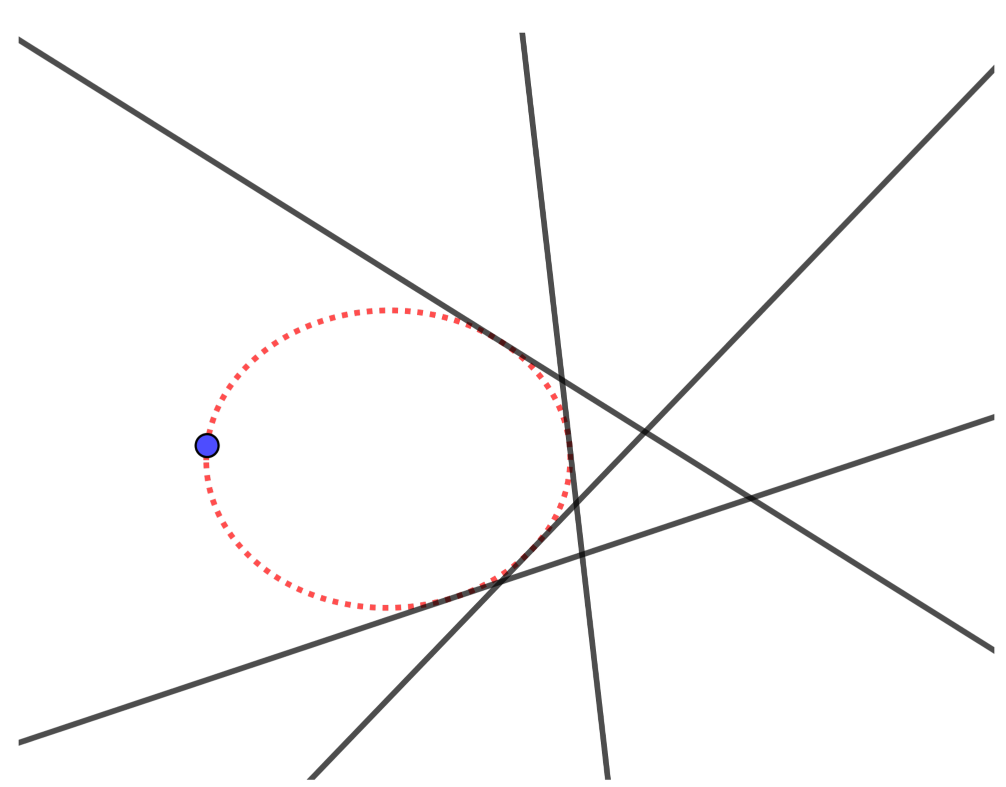
- $\phi(n,d)$ is the *maximum likelihood degree* of the *linear concentration model* defined by a generic d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{R}^n$.
- $\phi(n,d)$ is the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{C}^n$.
- $\phi(n,d)$ is the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2}-d$ given points and are tangent to $(d-1)$ given hyperplanes.

Complete quadrics

- The space Φ of complete quadrics is the closure of the image of

$$\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}\left(S^2\mathbb{C}^n\right)\times\mathbb{P}\left(S^2(\textstyle\bigwedge^2\mathbb{C}^n)\right)\times\cdots\times\mathbb{P}\left(S^2(\textstyle\bigwedge^{n-1}\mathbb{C}^n)\right)$$
$$M \mapsto (M, \wedge^2 M, \dots, \wedge^{n-1}M)$$

- Then $\phi(n,d)$ is the degree of the product $\mu_1^{d-1}\mu_{n-1}^{\binom{n+1}{2}-d}$ in the Chow ring (cohomology ring) $A(\Phi)$, where $\mu_i \in A^1(\Phi)$ is the pullback of the hyperplane class in $\mathbb{P}(S^2(\wedge^i\mathbb{C}^n))$.



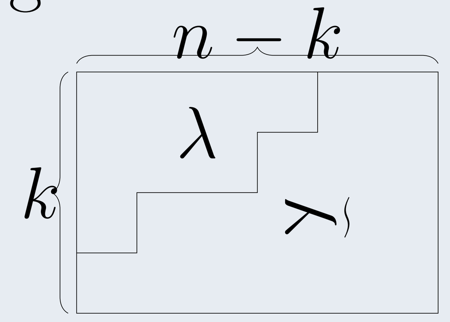
A formula for $\phi(n,d)$

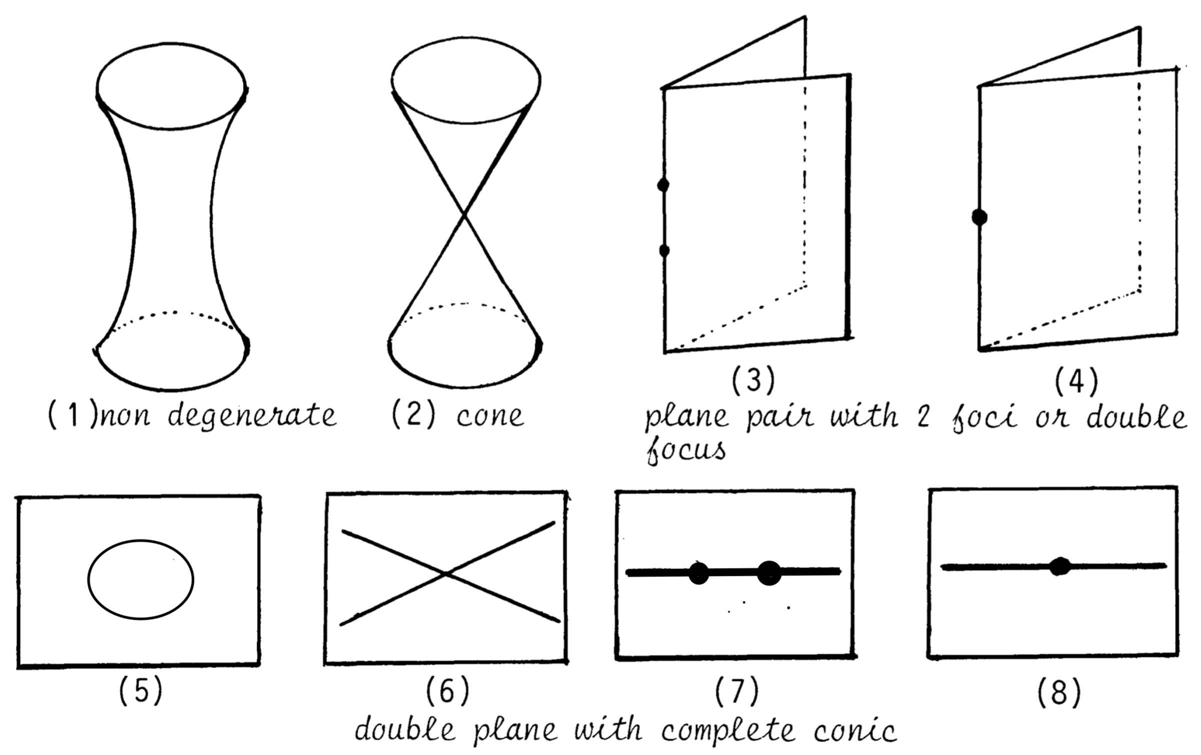
Define ψ_λ as the coefficients in the Schur decomposition of

$$H_\ell(x_i+x_j \mid 1\le i\le j\le k)=\sum_{\lambda\vdash\ell}\psi_\lambda S_\lambda(x_1,\dots,x_k),$$

where H_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. Then

$$\phi(n,d)=\frac{1}{n}\sum_{k=1}^{n}k\left(\sum_{\lambda}\psi_\lambda\psi_{\tilde\lambda}\right),\quad \text{where } \lambda\vdash d-\binom{k+1}{2},\quad \text{and}$$





Idea of the proof

- *Degeneration class* $\delta_k := [S_k] \in A^1(\Phi)$, where $S_k := \{(M_1,\dots,M_{n-1})\in\Phi \mid \mathrm{rk}(M_k)=1\}$.
- Using $2\mu_k=\mu_{k-1}+\delta_k+\mu_{k+1}$: suffices to compute $\mu_1^a\mu_{n-1}^b\delta_k$ for $a+b=\binom{n+1}{2}-2$.
- Pushforward along $\pi: S_k\to Gr(k,V)$: suffices to compute $\pi_*(\mu_1^a\mu_{n-1}^b)\in A(Gr(k,n))$.
- The Chow ring $A(Gr(k,n))$ is a quotient of the ring of symmetric functions.

Polynomiality and computations

Using our formula, and recursive relations between the ψ_λ, we can prove the following:

For fixed d, $\phi(n,d)$ is a polynomial in n, of degree $d-1$.

Moreover, we have an algorithm for computing these polynomials.
While previously, only the cases $d\le 5$ were known, our algorithm can compute $\phi(n,d)$ for $d\le 47$ in $\lesssim 5$ minutes. For instance:

$$\phi(n,18)=\frac{1}{355687428096000}(n-5)(n-4)(n-3)(n-2)(n-1)(3024902557n^{12}-111489409997n^{11}+1862235028288n^{10}-18676382506290n^9+125446336704681n^8-594987544526781n^7+2047718727437714n^6-5214795516381220n^5+10138037306327912n^4-15696938913831072n^3+18622763914779648n^2-12286614789872640n+2964061900800).$$

Based on joint work in progress with Laurent Manivel, Mateusz Michałek, Leonid Monin, Martin Vodicka, Andrzej Weber, and Jarosław, Wiśniweski

Complete quadrics and algebraic statistics

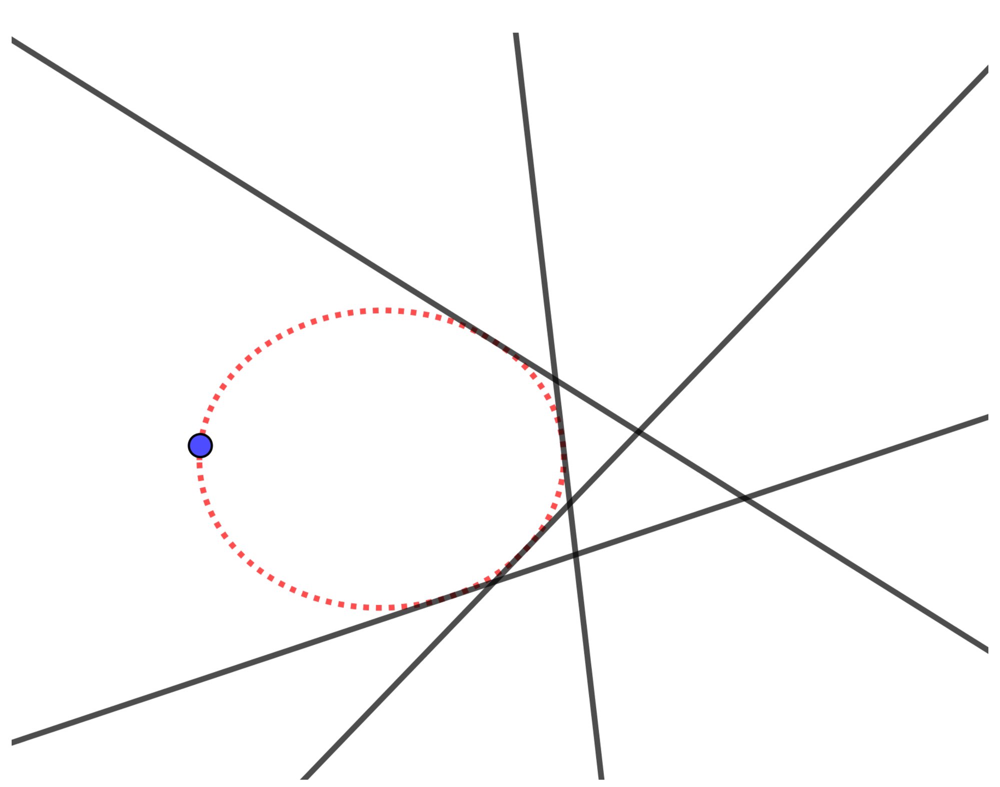
Tim Seynnaeve

Max Planck Institute for Mathematics in Sciences, Leipzig

Want to know more?
Check out my NASO talk, available at
https://www.mis.mpg.de/nlalg/seminars/naso.html.

Three equivalent definitions of the ML-degree $\phi(n,d)$

- $\phi(n,d)$ is the *maximum likelihood degree* of the *linear concentration model* defined by a generic d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{R}^n$.
- $\phi(n,d)$ is the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\mathrm{Sym}^2\,\mathbb{C}^n$.
- $\phi(n,d)$ is the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2}-d$ given points and are tangent to $(d-1)$ given hyperplanes.

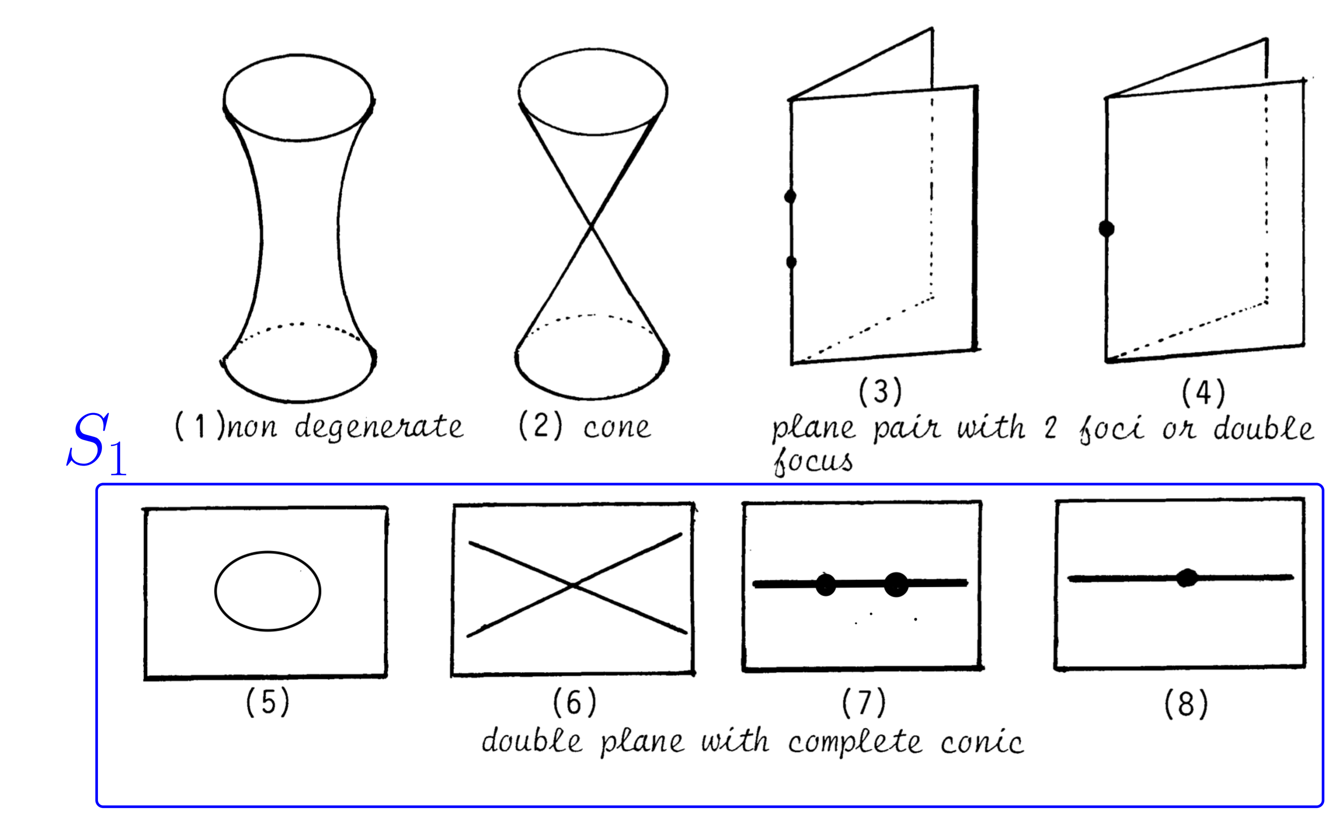


Complete quadrics

- The space Φ of complete quadrics is the closure of the image of

$$\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}\left(S^2\mathbb{C}^n\right) \times \mathbb{P}\left(S^2(\textstyle\bigwedge^2 \mathbb{C}^n)\right) \times \cdots \times \mathbb{P}\left(S^2(\textstyle\bigwedge^{n-1} \mathbb{C}^n)\right)$$
$$M \mapsto (M, \wedge^2 M, \ldots, \wedge^{n-1} M)$$

- Then $\phi(n,d)$ is the degree of the product $\mu_1^{d-1}\mu_{n-1}^{\binom{n+1}{2}-d}$ in the Chow ring (cohomology ring) $A(\Phi)$, where $\mu_i \in A^1(\Phi)$ is the pullback of the hyperplane class in $\mathbb{P}(S^2(\wedge^i \mathbb{C}^n))$.



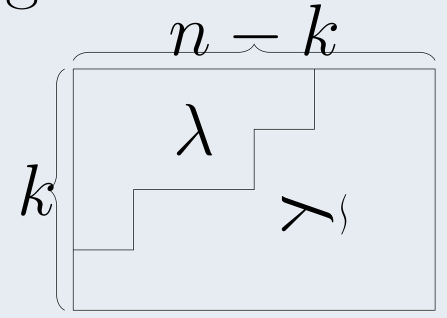
A formula for $\phi(n,d)$

Define ψ_λ as the coefficients in the Schur decomposition of

$$H_\ell(x_i + x_j \mid 1 \le i \le j \le k) = \sum_{\lambda \vdash \ell} \psi_\lambda S_\lambda(x_1, \ldots, x_k),$$

where H_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. Then

$$\boxed{\phi(n,d) = \frac{1}{n}\sum_{k=1}^{n} k \left(\sum_{\lambda} \psi_\lambda \psi_{\tilde{\lambda}}\right),} \quad \text{where } \lambda \vdash d - \binom{k+1}{2}, \quad \text{and } k$$



Idea of the proof

- *Degeneration class* $\delta_k := [S_k] \in A^1(\Phi)$, where $S_k := \{(M_1, \ldots, M_{n-1}) \in \Phi \mid \mathrm{rk}(M_k) = 1\}$.
- Using $2\mu_k = \mu_{k-1} + \delta_k + \mu_{k+1}$: suffices to compute $\mu_1^a \mu_{n-1}^b \delta_k$ for $a + b = \binom{n+1}{2} - 2$.
- Pushforward along $\pi : S_k \to Gr(k,V)$: suffices to compute $\pi_*(\mu_1^a \mu_{n-1}^b) \in A(Gr(k,n))$.
- The Chow ring $A(Gr(k,n))$ is a quotient of the ring of symmetric functions.

Polynomiality and computations

Using our formula, and recursive relations between the ψ_λ, we can prove the following:

For fixed d, $\phi(n,d)$ is a polynomial in n, of degree $d-1$.

Moreover, we have an algorithm for computing these polynomials.
While previously, only the cases $d \le 5$ were known, our algorithm can compute $\phi(n,d)$ for $d \le 47$ in $\lesssim 5$ minutes. For instance:

$$\phi(n,18) = \frac{1}{355687428096000}(n-5)(n-4)(n-3)(n-2)(n-1)(3024902557n^{12} - 111489409997n^{11} + 1862235028288n^{10} - 18676382506290n^9 + 125446336704681n^8 - 594987544526781n^7 + 2047718727437714n^6 - 5214795516381220n^5 + 10138037306327912n^4 - 15696938913831072n^3 + 18622763914779648n^2 - 12286614789872640n + 2964061900800).$$

Based on joint work in progress with Laurent Manivel, Mateusz Michałek, Leonid Monin, Martin Vodicka, Andrzej Weber, and Jarosław, Wiśniewski

Complete quadrics and algebraic statistics

Tim Seynnaeve

Max Planck Institute for Mathematics in Sciences, Leipzig

Want to know more?
Check out my NASO talk, available at
https://www.mis.mpg.de/nlalg/seminars/naso.html.

Three equivalent definitions of the ML-degree $\phi(n,d)$

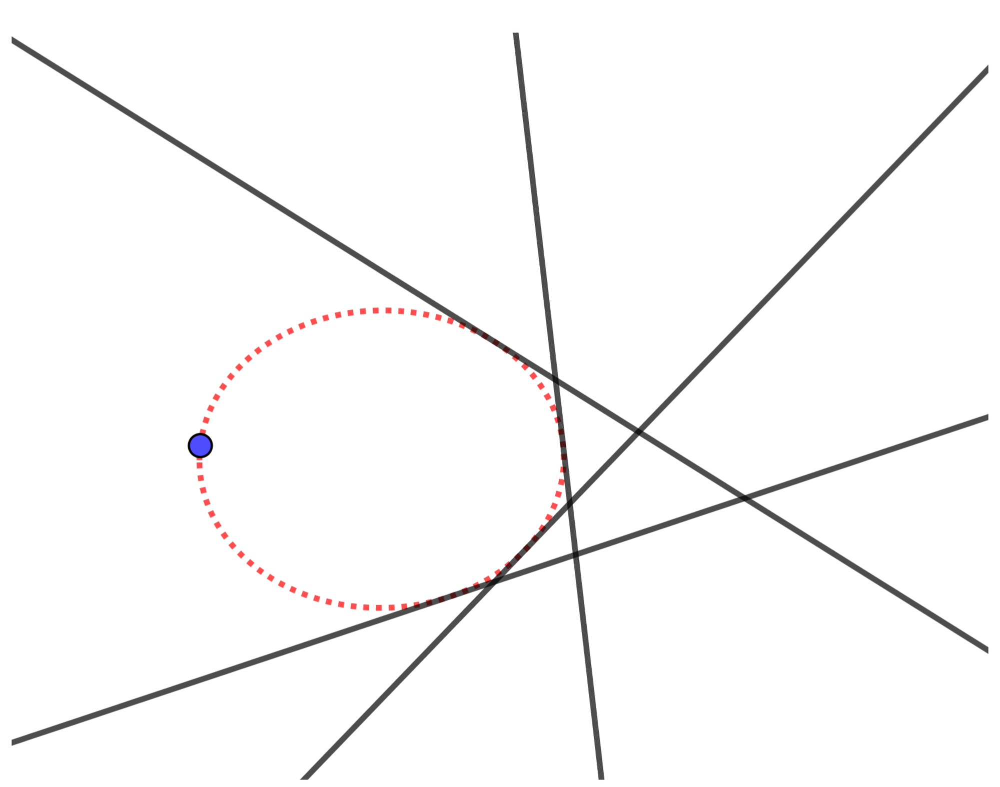
- $\phi(n,d)$ is the *maximum likelihood degree* of the *linear concentration model* defined by a generic d-dimensional linear subspace of $\operatorname{Sym}^2 \mathbb{R}^n$.
- $\phi(n,d)$ is the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\operatorname{Sym}^2 \mathbb{C}^n$.
- $\phi(n,d)$ is the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2} - d$ given points and are tangent to $(d-1)$ given hyperplanes.

Complete quadrics

- The space Φ of complete quadrics is the closure of the image of

$$\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}\left(S^2\mathbb{C}^n\right) \times \mathbb{P}\left(S^2(\textstyle\bigwedge^2 \mathbb{C}^n)\right) \times \cdots \times \mathbb{P}\left(S^2(\textstyle\bigwedge^{n-1} \mathbb{C}^n)\right)$$
$$M \mapsto (M, \wedge^2 M, \ldots, \wedge^{n-1} M)$$

- Then $\phi(n,d)$ is the degree of the product $\mu_1^{d-1}\mu_{n-1}^{\binom{n+1}{2}-d}$ in the Chow ring (cohomology ring) $A(\Phi)$, where $\mu_i \in A^1(\Phi)$ is the pullback of the hyperplane class in $\mathbb{P}(S^2(\wedge^i \mathbb{C}^n))$.



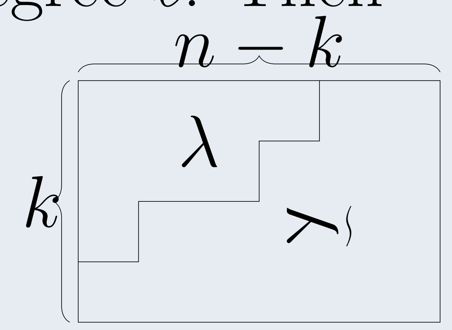
A formula for $\phi(n,d)$

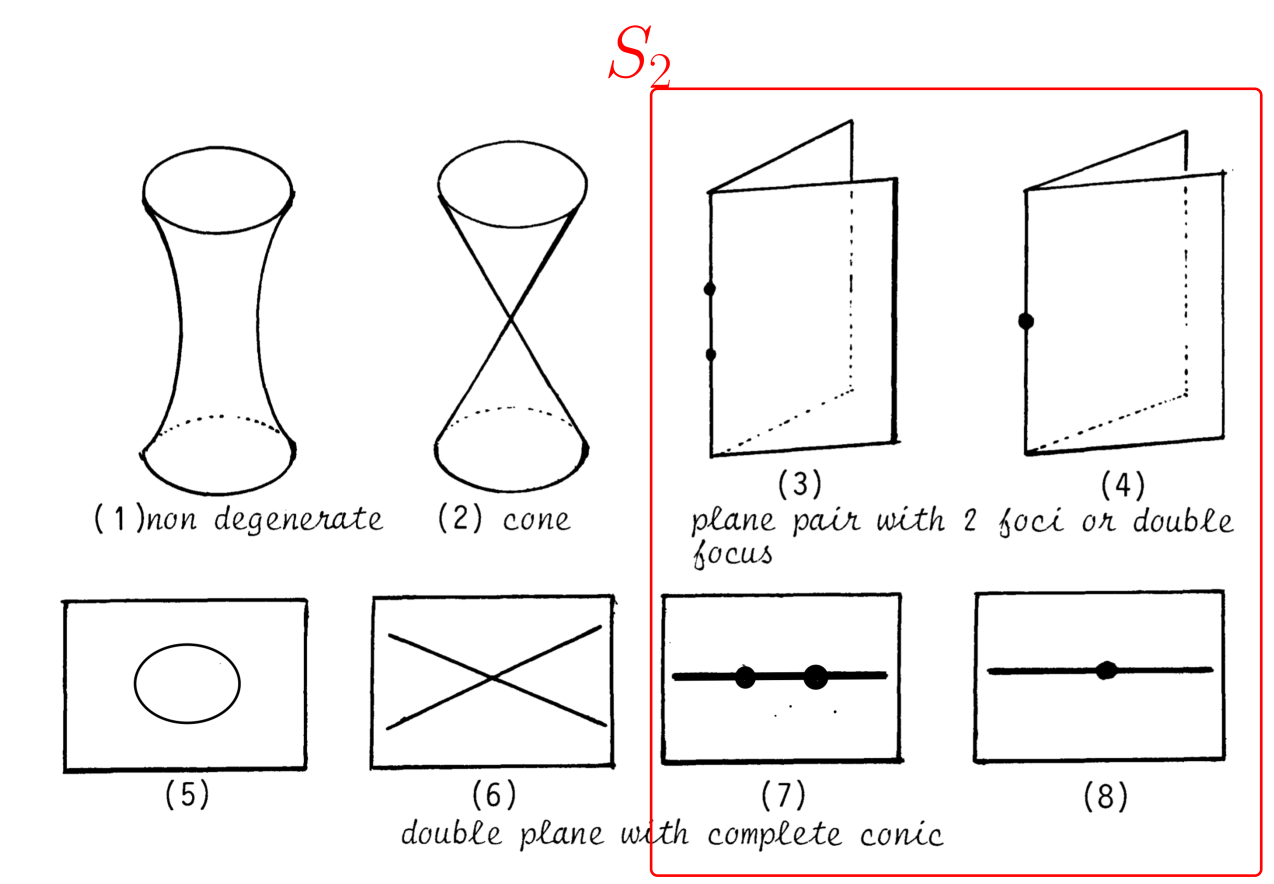
Define ψ_λ as the coefficients in the Schur decomposition of

$$H_\ell(x_i + x_j \mid 1 \le i \le j \le k) = \sum_{\lambda \vdash \ell} \psi_\lambda S_\lambda(x_1, \ldots, x_k),$$

where H_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. Then

$$\boxed{\phi(n,d) = \frac{1}{n} \sum_{k=1}^{n} k \left(\sum_\lambda \psi_\lambda \psi_{\tilde{\lambda}} \right),} \quad \text{where } \lambda \vdash d - \binom{k+1}{2}, \quad \text{and} \quad k$$





Idea of the proof

- *Degeneration class* $\delta_k := [S_k] \in A^1(\Phi)$, where $S_k := \{(M_1, \ldots, M_{n-1}) \in \Phi \mid \operatorname{rk}(M_k) = 1\}$.
- Using $2\mu_k = \mu_{k-1} + \delta_k + \mu_{k+1}$: suffices to compute $\mu_1^a \mu_{n-1}^b \delta_k$ for $a + b = \binom{n+1}{2} - 2$.
- Pushforward along $\pi : S_k \to Gr(k,V)$: suffices to compute $\pi_*(\mu_1^a \mu_{n-1}^b) \in A(Gr(k,n))$.
- The Chow ring $A(Gr(k,n))$ is a quotient of the ring of symmetric functions.

Polynomiality and computations

Using our formula, and recursive relations between the ψ_λ, we can prove the following:

For fixed d, $\phi(n,d)$ is a polynomial in n, of degree $d-1$.

Moreover, we have an algorithm for computing these polynomials.
While previously, only the cases $d \le 5$ were known, our algorithm can compute $\phi(n,d)$ for $d \le 47$ in $\lesssim 5$ minutes. For instance:

$$\phi(n,18) = \frac{1}{355687428096000}(n-5)(n-4)(n-3)(n-2)(n-1)(3024902557n^{12} - 111489409997n^{11} + 1862235028288n^{10} - 18676382506290n^9 + 125446336704681n^8 - 594987544526781n^7 + 2047718727437714n^6 - 5214795516381220n^5 + 10138037306327912n^4 - 15696938913831072n^3 + 18622763914779648n^2 - 12286614789872640n + 2964061900800).$$

Based on joint work in progress with Laurent Manivel, Mateusz Michałek, Leonid Monin, Martin Vodicka, Andrzej Weber, and Jarosław, Wiśniweski

Complete quadrics and algebraic statistics

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Tim Seynnaeve

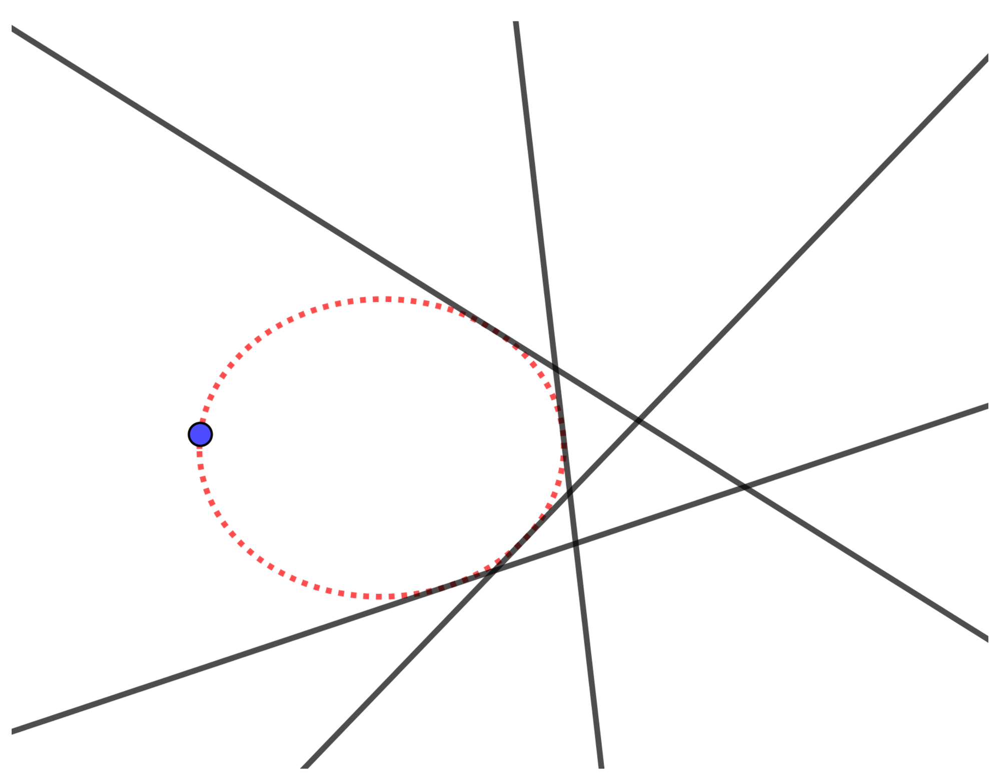
Max Planck Institute for Mathematics in Sciences, Leipzig

Want to know more?

Check out my NASO talk, available at https://www.mis.mpg.de/nlalg/seminars/naso.html.

Three equivalent definitions of the ML-degree $\phi(n,d)$

- $\phi(n,d)$ is the *maximum likelihood degree* of the *linear concentration model* defined by a generic d-dimensional linear subspace of $\operatorname{Sym}^2 \mathbb{R}^n$.
- $\phi(n,d)$ is the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\operatorname{Sym}^2 \mathbb{C}^n$.
- $\phi(n,d)$ is the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2} - d$ given points and are tangent to $(d-1)$ given hyperplanes.



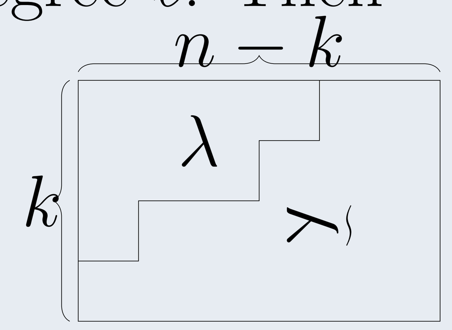
A formula for $\phi(n,d)$

Define ψ_λ as the coefficients in the Schur decomposition of

$$H_\ell(x_i + x_j \mid 1 \le i \le j \le k) = \sum_{\lambda \vdash \ell} \psi_\lambda S_\lambda(x_1, \ldots, x_k),$$

where H_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. Then

$$\boxed{\phi(n,d) = \frac{1}{n} \sum_{k=1}^{n} k \left(\sum_{\lambda} \psi_\lambda \psi_{\tilde{\lambda}} \right)}, \quad \text{where } \lambda \vdash d - \binom{k+1}{2}, \quad \text{and}$$



Idea of the proof

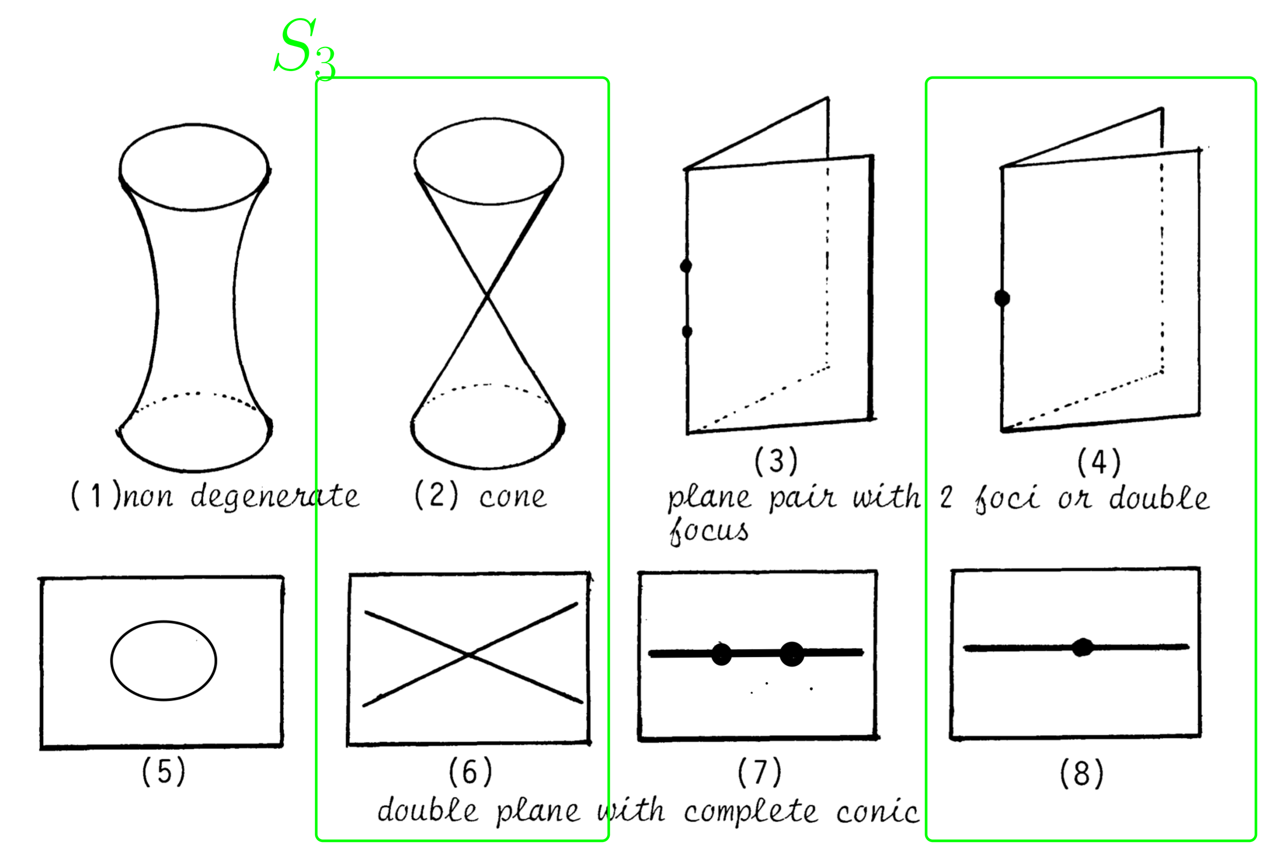
- *Degeneration class* $\delta_k := [S_k] \in A^1(\Phi)$, where $S_k := \{(M_1, \ldots, M_{n-1}) \in \Phi \mid \operatorname{rk}(M_k) = 1\}$.
- Using $2\mu_k = \mu_{k-1} + \delta_k + \mu_{k+1}$: suffices to compute $\mu_1^a \mu_{n-1}^b \delta_k$ for $a + b = \binom{n+1}{2} - 2$.
- Pushforward along $\pi : S_k \to Gr(k,V)$: suffices to compute $\pi_*(\mu_1^a \mu_{n-1}^b) \in A(Gr(k,n))$.
- The Chow ring $A(Gr(k,n))$ is a quotient of the ring of symmetric functions.

Complete quadrics

- The space Φ of complete quadrics is the closure of the image of

$$\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}\left(S^2\mathbb{C}^n\right) \times \mathbb{P}\left(S^2(\textstyle\bigwedge^2 \mathbb{C}^n)\right) \times \cdots \times \mathbb{P}\left(S^2(\textstyle\bigwedge^{n-1} \mathbb{C}^n)\right)$$
$$M \mapsto (M, \wedge^2 M, \ldots, \wedge^{n-1} M)$$

- Then $\phi(n,d)$ is the degree of the product $\mu_1^{d-1} \mu_{n-1}^{\binom{n+1}{2}-d}$ in the Chow ring (cohomology ring) $A(\Phi)$, where $\mu_i \in A^1(\Phi)$ is the pullback of the hyperplane class in $\mathbb{P}(S^2(\wedge^i \mathbb{C}^n))$.



Polynomiality and computations

Using our formula, and recursive relations between the ψ_λ, we can prove the following:

For fixed d, $\phi(n,d)$ is a polynomial in n, of degree $d-1$.

Moreover, we have an algorithm for computing these polynomials.
While previously, only the cases $d \le 5$ were known, our algorithm can compute $\phi(n,d)$ for $d \le 47$ in $\lesssim 5$ minutes. For instance:

$$\phi(n,18) = \frac{1}{355687428096000}(n-5)(n-4)(n-3)(n-2)(n-1)(3024902557n^{12} - 111489409997n^{11} + 1862235028288n^{10} - 18676382506290n^9 + 125446336704681n^8 - 594987544526781n^7 + 2047718727437714n^6 - 5214795516381220n^5 + 10138037306327912n^4 - 15696938913831072n^3 + 18622763914779648n^2 - 12286614789872640n + 2964061900800).$$

Based on joint work in progress with Laurent Manivel, Mateusz Michałek, Leonid Monin, Martin Vodicka, Andrzej Weber, and Jarosław, Wiśniweski