A tale of two polytopes 1: the bipermutahedron

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AlCoVe
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Part 1 is joint work with Graham Denham + June Huh (15-20).

Part 2 is joint work with Laura Escobar (20).
The plan

1. What is the bipermutahedral fan?
2. What is the bipermutahedron?
The permutahedral fan as a moduli space

**Permutahedral fan** $\Sigma_n$ in $N_n = \mathbb{R}^n / \mathbb{R}$:

Hyperplane arrangement $x_i = x_j$ for $i \neq j$ in $N_n$. 
The permutahedral fan as a moduli space

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Stratification: relative order

Strata: ordered set partitions $3\,28\,04\,1\,7\,569$
The bipermutahedral fan as a moduli space

**Bipermutahedral fan** $\Sigma_{n,n}$ in $\mathcal{N}_n \times \mathcal{N}_n$:

Moduli space: $n$-tuples of points in $\mathbb{R}^2$ (mod common translation)
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**Bipermutahedral fan** $\Sigma_{n,n}$ in $\mathcal{N}_n \times \mathcal{N}_n$:

Moduli space: $n$-tuples of points in $\mathbb{R}^2$ (mod common translation)

- $p_0$
- $p_1$
- $p_2$
- $p_3$
- $p_4$
- $p_5$

Stratification:
- draw lowest supporting $-45^\circ$ diagonal $\ell$
- record relative order of $x$ and $y$ projections onto $\ell$

Strata: **bisequences** 34|2|035|1|24|0
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![Diagram showing points $p_0, p_1, p_2, p_3, p_4, p_5$ arranged on a line with supporting $-45^\circ$ diagonal $\ell$]

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The bipermutahedral fan as a moduli space

**Bipermutahedral fan** $\Sigma_{n,n}$ in $\mathbb{N}_n \times \mathbb{N}_n$:
Moduli space: $n$-tuples of points in $\mathbb{R}^2$ (mod common translation)

Strata: **bisequences** on $[n]$

Sequences $\mathcal{B} = B_1 | \cdots | B_m$ such that
- each number appears once or twice,
- some number appears exactly once.

Ex: $34|2|035|1|24|0$
The bipermutahedron

Permutahedral fan $\Sigma_n$:
Normal fan of the permutahedron $\Pi_n$.

Bipermutahedral fan $\Sigma_{n,n}$:
Normal fan of the bipermutahedron $\Pi_{n,n}$.

For the precise construction, see [Ardila-Denham-Huh 2020].
Combinatorial structure of the bipermutahedron

- **faces**: bisequences $12|45|4|235$

- **vertices**: bipermutations $1|5|4|1|3|4|2|5|3$. (one number appears once, others twice) $(2n)!/2^n$

- **facets**: bisubsets $1245|235$ $(S, T \neq \emptyset$, not both $[n]$, with $S \cup T = [n])$ $3^n - 3$
The $h$-vector of the bipermutahedron

The bipermutahedron is simple; consider its $h$-polynomial:

$$h_n(x) = h_0(\Pi_{n,n}) + h_1(\Pi_{n,n})x + \cdots + h_{2n-2}(\Pi_{n,n})x^{2n-2}$$

We call it the biEulerian polynomial.
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\[ h_i(\Pi_{n,n}) = \text{# of bipermutations of } [n] \text{ with } i \text{ descents.} \]
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- The biEulerian polynomial is given by

$$\frac{h_n(x)}{(1-x)^{2n+1}} = \sum_{k \geq 0} \binom{k+2}{2}^n x^k$$
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- All roots of the biEulerian polynomial are real and negative.
- The $h$-vector of the bipermutahedron is log-concave.
Origin story: the geometry of matroids

Matroids capture the combinatorial essence of independence.

Prototypical example:
\[ E = \text{set of vectors in a vector space } V \]
\[ \mathcal{F} = \{ \text{spans of subsets of } E \} \]
(a poset under \( \subseteq \))

**Definition.** A matroid \((E, \mathcal{F})\) is
- a set \(E\) and
- a collection \(\mathcal{F}\) of subsets of \(E\) satisfying some axioms.
Numerical invariants

Given a matroid $M$,

$n = \text{number of elements}$

$r = \text{rank} = \text{height of poset}$

$f$-vector $= |\text{coeffs}|$ of $\chi_M(q)$

$h$-vector $= |\text{coeffs}|$ of $\chi_M(q+1)$

Ex: $n = 5 \quad r = 3 \quad f = (1, 4, 5, 2) \quad h = (1, 1, 0, 0)$

Theorem.

1. [Adiprasito-Huh-Katz ’15] $f_0, f_1, \ldots, f_r$ is log-concave.
   Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

2. [Ardila-Denham-Huh ’20] $h_0, h_1, \ldots, h_r$ is log-concave.
   Conjectured by Brylawski 82, Dawson 83, Hibi 89.
Log-concavity of $f$-vector: geometry of matroids
[Adiprasito–Huh–Katz 15]

1. Use the **Bergman fan** $\Sigma_M$ as a geometric model for $M$.
   $(r - 1)$-dim fan in $N_n$,
   $\text{Supp}(\Sigma_M) = \text{Trop}(M)$ \ [FA-Klivans 06]

2. Find classes $\alpha, \beta$ in the Chow ring $A^\bullet(\Sigma_M)$ with

   $\alpha^{r-i} \beta^i = f_i \quad (1 \leq i \leq r)$

3. Prove the Hodge-Riemann relations for the fan $\Sigma_M$.
   They imply $(\alpha^{r-i} \beta^i : 0 \leq i \leq r)$ is log-concave.
Log-conc of $h$-vector: Lagrangian geom of matroids

[Ardila–Denham–Huh 20]

1. Use the **conormal fan** $\Sigma_{M, M^\perp}$ as a geometric model for $M$. 
   $(n - 2)$-dim fan in $\mathbb{N}_n \times \mathbb{N}_n$

2. Find classes $\gamma, \delta$ in the Chow ring $A^\bullet(\Sigma_{M, M^\perp})$ with
   $$
   \gamma^i \delta^{n-2-i} = h_{r-i} \quad (1 \leq i \leq r)
   $$

3. Prove the Hodge-Riemann relations for the fan $\Sigma_{M, M^\perp}$. 
   They imply $(\gamma^i \delta^{n-2-i} : 0 \leq i \leq r)$ is log-concave.
How to define the conormal fan $\Sigma_{\mathcal{M},\mathcal{M}^\perp}$?

Varchenko’s **critical set varieties** offer hints/requirements:

1. Support$(\Sigma_{\mathcal{M},\mathcal{M}^\perp})$ “should be" $\text{Trop}(\mathcal{M}) \times \text{Trop}(\mathcal{M}^\perp)$.
   Tropical analog of conormal bundle.

2. $\Sigma_{\mathcal{M},\mathcal{M}^\perp}$ “should be" simplicial, so the Chow ring is tractable.
   Try: $\Sigma_{\mathcal{M},\mathcal{M}^\perp} = \Sigma_\mathcal{M} \times \Sigma_{\mathcal{M}^\perp}$?
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   Try: $\Sigma_{M, M^\perp} = \Sigma_M \times \Sigma_{M^\perp}$?

3. There “should be" classes $\gamma$ and $\delta$ with $\gamma^i \delta^{n-2-i} = h_{r-i}$ (*)
   - $\gamma$ “should be" the pullback of $\alpha$ along
     $\pi : \Sigma_M \times \Sigma_{M^\perp} \to \Sigma_M$, $\pi(x, y) = x$
   - $\delta$ "should be" the pullback of $\alpha$ along
     $\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_n$, $\sigma(x, y) = x + y$
     where $\Delta_n$ is the normal fan of the standard simplex.

- Geometry predicts (*), prove it algebro-combinatorially.
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3. There “should be" classes $\gamma$ and $\delta$ with $\gamma^i \delta^{n-2-i} = h_{r-i}$ $(\ast)$
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   - Geometry predicts $(\ast)$, prove it algebro-combinatorially.

**Problem:** $\sigma$ is not a map of fans!
How to define the conormal fan $\Sigma_{M,M^\perp}$?

**Problem:** $\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_E$, $\sigma(x, y) = x + y$ not a map of fans!

**Solution:** Subdivide $\Sigma_M \times \Sigma_{M^\perp}$ so $\sigma$ is a map of fans. Pero how?
How to define the conormal fan $\Sigma_{M, M^\perp}$?

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Idea: Do it simultaneously for all matroids on $E$.

[FA – Klivans 06]
Permutahedral fan $\Sigma_E$ resolved this issue for all Bergman fans:
$$\Sigma_M := \text{Trop}(M) \cap \Sigma_E$$

[FA – Denham – Huh 20]
**Bipermutahedral fan** $\Sigma_{E, E}$ resolves this for all conormal fans:
$$\Sigma_{M, M^\perp} := (\text{Trop}(M) \times \text{Trop}(M^\perp)) \cap \Sigma_{E, E}$$
How to define the bipermutahedral fan?

What do we want?
A nice complete fan $\Sigma$ in $\mathbb{N}_n \times \mathbb{N}_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n$, $\pi(x, y) = x$ is a map of fans
b. $\pi_2 : \Sigma \to \Sigma_n$, $\pi(x, y) = y$ is a map of fans
c. $\sigma : \Sigma \to \Delta_n$, $\sigma(x, y) = x + y$ is a map of fans

where $\Sigma_n = \text{braid fan}$ and $\Delta_n = \text{fan of } \mathbb{P}^{n-1}$.

d. It is the normal fan of a polytope.

Try 1: $\Sigma = \text{coarsest refinement of } \Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$. 
How to define the bipermutahedral fan?

What do we want?

A **nice** complete fan $\Sigma$ in $N_n \times N_n$ such that:

- $\pi_1 : \Sigma \to \Sigma_n$, $\pi(x, y) = x$ is a map of fans
- $\pi_2 : \Sigma \to \Sigma_n$, $\pi(x, y) = y$ is a map of fans
- $\sigma : \Sigma \to \Delta_n$, $\sigma(x, y) = x + y$ is a map of fans

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**Try 1:** $\Sigma = \text{coarsest refinement of } \Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

This is the **harmonic fan/polytope** $H_{n,n}$ of Laura Escobar’s talk.

**Good news:** It has all these properties + beautiful combinatorics.

**Bad news:** It is not simplicial. How to compute in its Chow ring?
How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

**Try 1:** $H_{n,n} = $ coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

**Try 2:** $\Sigma_{n,n} = $ **nice polytopal simplicial** refinement of $H_{n,n}$.
How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

**Try 1:** \(H_{n,n} = \) coarsest refinement of \(\Sigma_n \times \Sigma_n\) and \(\sigma^{-1}(\Delta_n)\).

**Try 2:** \(\Sigma_{n,n} = \) nice polytopal simplicial refinement of \(H_{n,n}\).

How do we find it? It’s more of an art than a science...

The **bipermutohedral fan** \(\Sigma_{n,n}\) is the nicest one we could find.
muchas gracias
