Combinatorics of quadratic spaces over finite fields

Motivation

One of the surprising phenomenon about the q-binomial coefficient is the fact that there are the q-analogues of the binomial coefficient. There is a well-known interesting relationship between the binomial coefficients and the q-binomial coefficients as follows:

	Field with one element	$\mathbb{F}_q$ (q-analogues)
object	$[n] = \{1, 2, \cdots, n\}$	$\mathbb{F}_q^n$
subobject	a k set in $[n]$	a k-dimensional subspace of $\mathbb{F}_q^n$
bracket	n	the number of lines in $\mathbb{F}_q^n$
factorial	$n!$	$[n]_q!$
poset	B_n	$L_n(q)$
group	$\lvert S_n\rvert = n!$	$\lvert GL(n,q)\rvert = q^{n(n-1)/2}(q-1)^n\,[n]_q!$
flag	flags in $[n]$	flags in $\mathbb{F}_q^n$
binomial coefficient	$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \left\lvert\frac{S_n}{S_k \times S_{n-k}}\right\rvert$	$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![(n-k)]_q!} = \left\lvert\frac{GL(n,q)}{\left(\begin{smallmatrix} A & C \\ \mathbf{0} & B\end{smallmatrix}\right)}\right\rvert$
connection	$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k}$	

Table 1: Example of Field with one element analogues.

In this project, we add one more bit of structure called a quadratic form on $\mathbb{F}_q^n$. We consider the quadratic form dot_n on $\mathbb{F}_q^n$ given by

$$\mathrm{dot}_n(\mathbf{x}) := x_1^2 + \cdots + x_n^2 \quad \text{for any } \mathbf{x} = (x_1, x_2, ..., x_n) \text{ in } \mathbb{F}_q^n.$$

The main goal is to define the **dot-analogues** of the q-binomial coefficients, and to study related combinatorics listed in the last column of Table 2.

	q-analogues	dot-analogues
space	$\mathbb{F}_q^n$	$(\mathbb{F}_q^n, \mathrm{dot}_n)$
subspace	a k-dimensional subspace of $\mathbb{F}_q^n$	a dot_k-subspace of dot_n
bracket	the number of lines in $\mathbb{F}_q^n$	the number of spacelike lines in $(\mathbb{F}_q^n, \mathrm{dot}_n)$
factorial	$[n]_q!$	$[n]_d!$
poset	$L_n(q)$	$E_n(q)$
group	$\lvert GL(n,q)\rvert = q^{n(n-1)/2}(q-1)^n\,[n]_q!$	$\lvert O(n,q)\rvert = 2^n\,[n]_d!$
flag	flags in $\mathbb{F}_q^n$	Euclidean flags in $(\mathbb{F}_q^n, \mathrm{dot}_n)$
binomial coefficient	$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![(n-k)]_q!} = \left\lvert\frac{GL(n,q)}{\left(\begin{smallmatrix} A & C \\ \mathbf{0} & B\end{smallmatrix}\right)}\right\rvert$	$\binom{n}{k}_d = \frac{[n]_d!}{[k]_d![(n-k)]_d!} = \left\lvert\frac{O(n,q)}{O(k,q)\times O(n-k,q)}\right\rvert$
connection	$\lim_{q\to 1}\binom{n}{k}_d =$ double factorial binomial coefficients or 0	

Table 2: The q-analogues and the dot-analogues.

Teminologies.

- A k-dimensional quadratic subspace W in $(\mathbb{F}_q^n, dot_n)$ is called **dot_k-subspace** if $W|_{\mathrm{dot}_n} \simeq \mathrm{dot}_k$.
- Let us denote $\mathrm{dot}_{k,n}$ by the set of dot_k-subspaces in a fixed dot_n.
- Let us call elements in $\mathrm{dot}_{1,n}$ **spacelike lines**.

The dot-binomial coefficients

For any n and k, we define

- $[k]_d := \lvert\mathrm{dot}_{1,k}\rvert$;
- $[n]_d! := [n]_d \cdots [1]_d$;
- $\binom{n}{k}_d := \lvert\mathrm{dot}_{k,n}\rvert = \frac{[n!]_d}{[k!]_d[(n-k)!]_d}$.

We call these **dot-analogs**. In particular, we call $\binom{n}{k}_d$ **dot-binomial coefficients**. We adopt the convention that $\lvert\mathrm{dot}_{1,0}\rvert := 1$. In other words,

$$\binom{n}{k}_d = \frac{\lvert\mathrm{dot}_{1,n}\rvert\lvert\mathrm{dot}_{1,n-1}\rvert\cdots\lvert\mathrm{dot}_{1,n-k+1}\rvert}{\lvert\mathrm{dot}_{1,k}\rvert\cdots\lvert\mathrm{dot}_{1,1}\rvert},$$

For example,

$$\begin{aligned}\binom{13}{5}_d &= \frac{(q^{12}+q^6)(q^{11}-q^5)(q^{10}+q^5)(q^9-q^4)(q^8+q^4)}{2(q^4+q^2)(q^3-q)(q^2+q)(q-1)} \\ &= \frac{1}{2}q^{20}(q^4-q^2+1)(q^4+q^2+1)(q^4-q^3+q^2-q+1) \\ &\quad \cdot (q^4+q^3+q^2+q+1)(q^4+1).\end{aligned}$$

Thus $\binom{13}{5}_d$ is a polynomial of degree 40 in q.

Question. How to count $\lvert\mathrm{dot}_{1,n}\rvert$?

In $(\mathbb{F}_q^n, x_1^2 + x_2^2 + \cdots + x_n^2)$, the number of spacelike lines is:

<table>
<tr><th>Spacelike</th><th>$q \equiv 1 \bmod 4$</th><th>$q \equiv 3 \bmod 4$</th></tr>
<tr><td>$n = 4k+3$</td><td rowspan="2">$\frac{q^{n-1}+q^{\frac{n-1}{2}}}{2}$</td><td>$\frac{q^{n-1}-q^{\frac{n-1}{2}}}{2}$</td></tr>
<tr><td>$n = 4k+1$</td><td>$\frac{q^{n-1}+q^{\frac{n-1}{2}}}{2}$</td></tr>
<tr><td>$n = 4k+2$</td><td rowspan="2">$\frac{q^{n-1}-q^{\frac{n-2}{2}}}{2}$</td><td>$\frac{q^{n-1}+q^{\frac{n-2}{2}}}{2}$</td></tr>
<tr><td>$n = 4k$</td><td>$\frac{q^{n-1}-q^{\frac{n-2}{2}}}{2}$</td></tr>
</table>

Table 3: The number of spacelike lines in dot_n.

Properties. The dot-binomial coefficients satisfy

- $\binom{n}{k}_d = \binom{n-1}{k-1}_d + \frac{[n]_d - [k]_d}{[n-k]_d}\binom{n-1}{k}_d$,
- For example, if n, k are odd,

$$\lim_{q\to 1}\binom{n}{k}_d = \left(\!\binom{n-1}{k-1}\!\right) = \binom{(n-1)/2}{(k-1)/2}.$$

- rational in q.

The Euclidean Posets

- Let us define a poset $E_n(q) := (\mathrm{dot}_{k,n}, \subset)$.
- Let us call it the **Euclidean poset**.
- We do not consider the empty set to be a subspace.
- We consider the zero space as the least element of the Euclidean poset.

Properties. The Euclidean posets $E_n(q)$ satisfy the following:

1. rank-symmetric: $\binom{n}{k}_d = \binom{n}{n-k}_d$,
2. rank-unimodal: $\exists\, j$ s.t $\binom{n}{0}_d \le \binom{n}{1}_d \le \cdots \le \binom{n}{j}_d \ge \cdots \ge \binom{n}{n}_d$,
3. log-concave: $\binom{n}{k}_d^2 \ge \binom{n}{k-1}_d\binom{n}{k+1}_d$,
4. Sperner:
   $\max\{\lvert A\rvert \mid A \text{ is an antichain of } E_n(q)\} = \max\{\lvert E_n(q)_k\rvert \mid 0 \le k \le n\}$.
5. The number of maximal Euclidean flags in $E_n(q)$ is $[n]_d! = [n]_d\,[n-1]_d \cdots [1]_d$

Applications from Graph Theory

Let us construct graphs with the vertex set V= dot_k-subspaces, and the edge set may be determined by

$$v \sim w \quad \Leftrightarrow \quad (1)\ v \subset w^{\perp} \text{ or } (2)\ v \subset w.$$

(1) Let $X, Y \subset V$ and $k < n/3$. If $\lvert X\rvert\lvert Y\rvert > (1+o(1))q^{nk}$, then there exist edges between X and Y. In other words, there exists dot_k-subspaces in X and Y which are orthogonal each other.

(2) We obtain bounds for the number of incidences $I(\mathcal{K}, \mathcal{H})$ between a collection $\mathcal{K}$ of dot_k-subspaces and a collection $\mathcal{H}$ of dot_h-subspaces when $h \ge 4k-4$, which is given by

$$\left\lvert I(\mathcal{K},\mathcal{H}) - \frac{\lvert\mathcal{K}\rvert\lvert\mathcal{H}\rvert}{q^{k(n-h)}}\right\rvert \lesssim q^{\frac{k(2h-n-2k+4)+h(n-h-1)-2}{2}}\sqrt{\lvert\mathcal{K}\rvert\lvert\mathcal{H}\rvert}.$$

References

[1] S. Yoo, *Combinatorics of quadratic spaces over finite fields*, preprint (2019).

[2] S. Yoo, *Graphs associated with orthgonal collection of k-planes over finite fields*, preprint (2020).

[3] S. Yoo, *Incidences between quadratic spaces over finite fields*, preprint (2020).