Some shelling orders are better than others

José Alejandro Samper

Max-Planck Institute for Mathematics in the Sciences
$\downarrow$
PUC Chile

June 16, 2020

AICoVE

Join work with

Alex Heaton
MPI / TU Berlin

Some shelling orders are better than others

Let K be a pure polytopal/simplicial complex. A shelling order is an organized way to build K by adding one facet at a time.

Some shelling orders are better than others

Let K be a pure polytopal/simplicial complex. A shelling order is an organized way to build K by adding one facet at a time.

They are used for varied reasons, among others, shelling orders allow us to compute:

Some shelling orders are better than others

Let K be a pure polytopal/simplicial complex. A shelling order is an organized way to build K by adding one facet at a time.

They are used for varied reasons, among others, shelling orders allow us to compute:

- Topological invariants, i.e homotopy type, Euler Characteristic.
- Combinatorial invariants, i.e Dehn-Sommerville equations.
- Algebraic invariants, i.e Cohen-Macaulayness.

Some shelling orders are better than others

Let K be a pure polytopal/simplicial complex. A shelling order is an organized way to build K by adding one facet at a time.

They are used for varied reasons, among others, shelling orders allow us to compute:

- Topological invariants, i.e homotopy type, Euler Characteristic.
- Combinatorial invariants, i.e Dehn-Sommerville equations.
- Algebraic invariants, i.e Cohen-Macaulayness.

Vague question

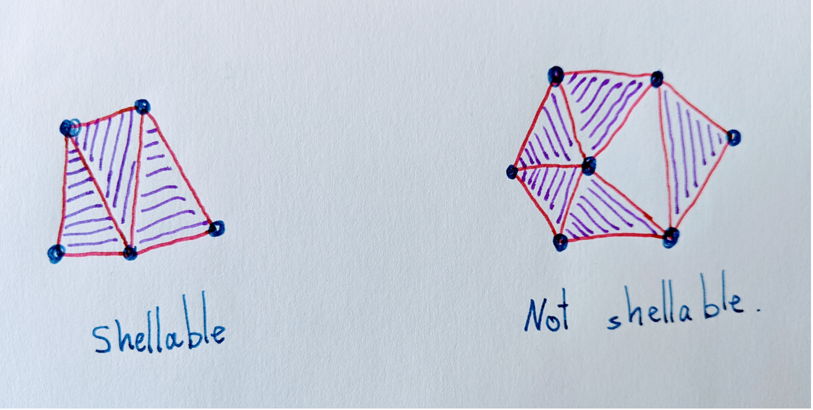
Is it enough to know that a complex is shellable? Or are some shelling orders better than others?

Warm up

Pure Simplicial Complex: E finite set, $\Delta \subseteq 2^E$, s.t $F \in \Delta$ and $G \subseteq F$ implies $G \in \Delta$, and all maximal elements have same size.

Warm up

Pure Simplicial Complex: E finite set, $\Delta \subseteq 2^E$, s.t $F \in \Delta$ and $G \subseteq F$ implies $G \in \Delta$, and all maximal elements have same size.



Shelling orders of simplicial complexes

Definition

A pure simplicial complex Δ is said to be *shellable* if there is an order $F_1, \ldots, F_k$ of the facets of Δ such that for each j, there is a unique minimal subset $\mathcal{R}(F_j)$ of F_j not contained in F_i for any $i < j$.

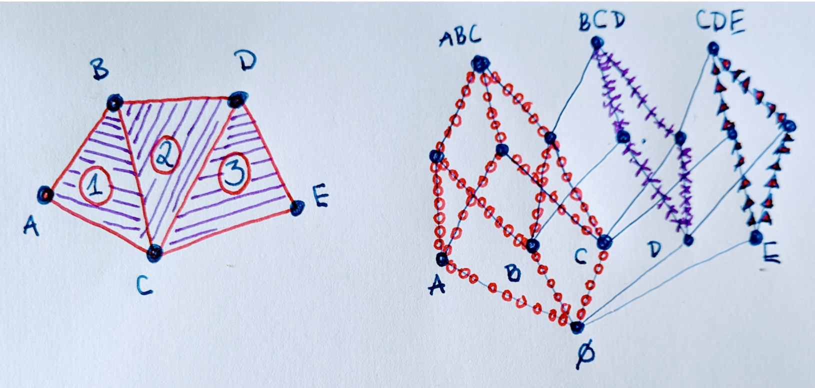
Shelling orders of simplicial complexes

Definition

A pure simplicial complex Δ is said to be *shellable* if there is an order $F_1, \ldots, F_k$ of the facets of Δ such that for each j, there is a unique minimal subset $\mathcal{R}(F_j)$ of F_j not contained in F_i for any $i < j$.

Such an order is called a shelling order. The special subset of F_j is called the restriction set of F_j in the shelling.

Example



Quantitative vs. qualitative properties of shelling orders

- **Quantitative aspects:** h-vector: $(h_0, ..., h_d)$ where h_j is the number of facets with $|\mathcal{R}(F)| = j$.

  Heavily studied:
  - h_d: homotopy type.
  - Proves Cohen-Macaulaynes.
  - Compute Hilbert series of SR-ring.

Quantitative vs. qualitative properties of shelling orders

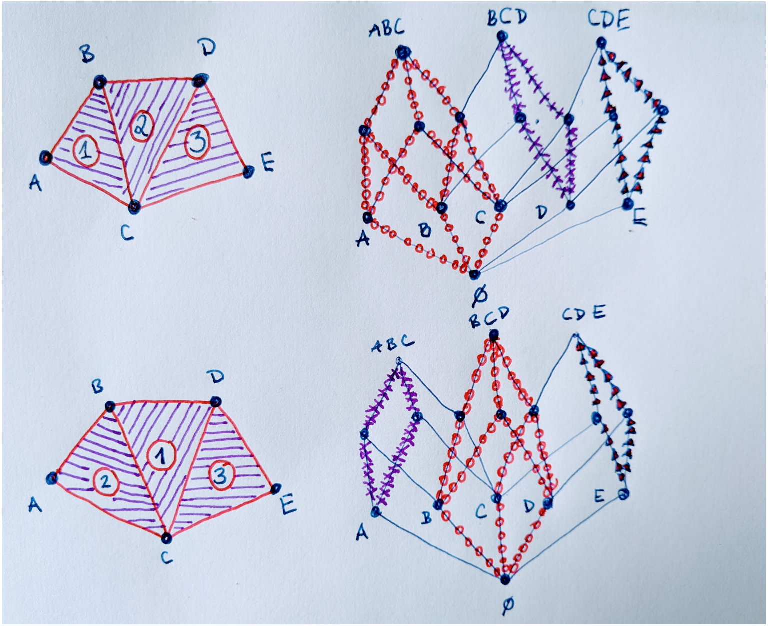
- **Quantitative aspects:** h-vector: $(h_0, ..., h_d)$ where h_j is the number of facets with $|\mathcal{R}(F)| = j$.

  Heavily studied:

  - h_d: homotopy type.
  - Proves Cohen-Macaulaynes.
  - Compute Hilbert series of SR-ring.

- **Qualitative aspects:** Set system $\{\mathcal{R}(F) \mid F \text{ facet}\}$. Partially ordered sets, hypergraphs. Much less is known!

Example



Yet another definition of matroids

Definition (A la Björner)

Matroid: pair $(E, \mathcal{I})$ with $\mathcal{I} \subset 2^E$ simplicial complex such that:

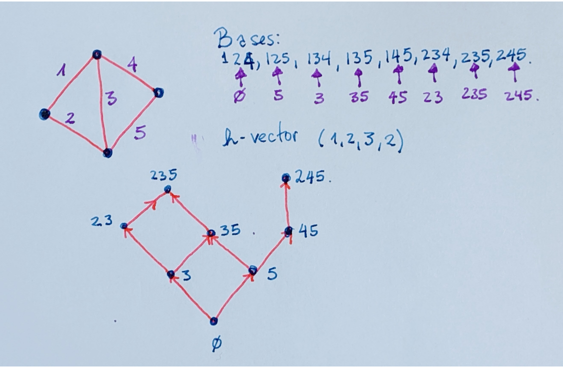
- For every order $<$ of E, the induced lexicographic order on the facets(bases) is a shelling order.

Yet another definition of matroids

Definition (A la Björner)

Matroid: pair $(E, \mathcal{I})$ with $\mathcal{I} \subset 2^E$ simplicial complex such that:

- For every order $<$ of E, the induced lexicographic order on the facets(bases) is a shelling order.



An old conjecture

Conjecture (Stanley 77)

The h-vector of a matroid is a pure O-sequence.

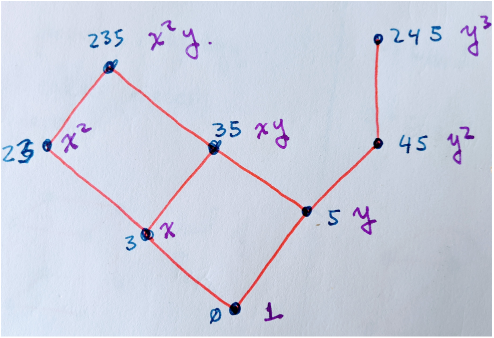
I.e there is a collection of monomials, closed under dibisibility, with h_i monomials of degree i and all maximal monomials of the same degree.

An old conjecture

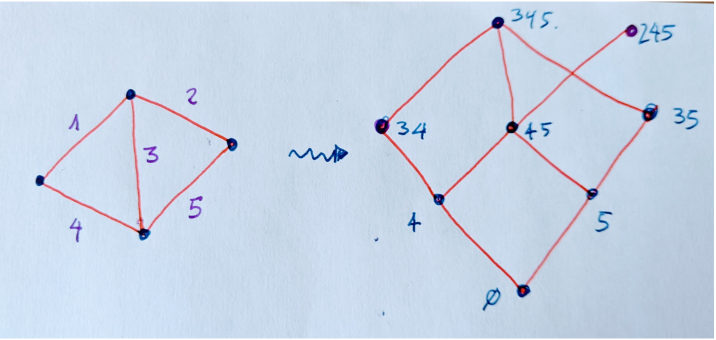
Conjecture (Stanley 77)

The h-vector of a matroid is a pure O-sequence.

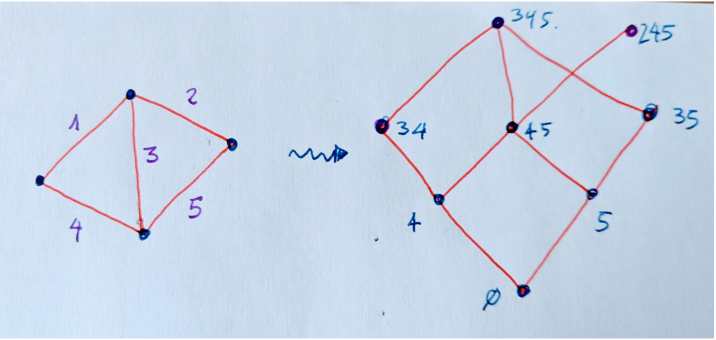
I.e there is a collection of monomials, closed under dibisibility, with h_i monomials of degree i and all maximal monomials of the same degree.



Unfortunately, the trick fails often.



Unfortunately, the trick fails often.



In other words, the first shelling order is better (relative to our problem).

Some facts:

- Dall (2017) proved that there are many ordered matroids for wich this works. (Internally perfect matroids). But some matroids admit no order at all!

Some facts:

- Dall (2017) proved that there are many ordered matroids for wich this works. (Internally perfect matroids). But some matroids admit no order at all!
- The restriction sets poset is similar to a divisibility poset. Some structural results (Dawson 84, Las Vergnas 01) guarantee purity (when it works).

Some facts:

- Dall (2017) proved that there are many ordered matroids for wich this works. (Internally perfect matroids). But some matroids admit no order at all!
- The restriction sets poset is similar to a divisibility poset. Some structural results (Dawson 84, Las Vergnas 01) guarantee purity (when it works).
- In small cases, it is easy to 'fix' the shelling order by slightly changing the order of some bases.

New idea: geometry may be useful

- **Matroid polytope:** $P_M \subset \mathbb{R}^E$ convex hull of caracteristic vectors of bases.

New idea: geometry may be useful

- **Matroid polytope:** $P_M \subset \mathbb{R}^E$ convex hull of caracteristic vectors of bases.

- **Dual matroid polytope:** P_M^* has facets in correspondence with bases of M.

New idea: geometry may be useful

- **Matroid polytope:** $P_M \subset \mathbb{R}^E$ convex hull of caracteristic vectors of bases.

- **Dual matroid polytope:** P_M^* has facets in correspondence with bases of M.

- **Nice fact:** Polytopes are always shellable (though a finer definition is needed).

New idea: geometry may be useful

- **Matroid polytope:** $P_M \subset \mathbb{R}^E$ convex hull of caracteristic vectors of bases.
- **Dual matroid polytope:** P_M^* has facets in correspondence with bases of M.
- **Nice fact:** Polytopes are always shellable (though a finer definition is needed).
- **Line shellings of Bruggesser and Mani:** Many shellings of P_M^* can be obtained by ordering vertices of P_M according to a linear functional.

New tool

Theorem (Heaton, S. 2020)

Every shelling order of P_M^* induces a shelling order of $\mathcal{I}$.

New tool

Theorem (Heaton, S. 2020)

Every shelling order of P_M^* induces a shelling order of $\mathcal{I}$.

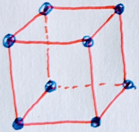
Theorem (Heaton, S. 2020)

Line shellings of P_M^* recover the theory of internal activity for $\mathcal{I}$.
I.e Björner's characterization is a shadow of a geometric fact.

The big picture

P_M^*

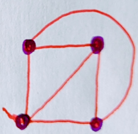
- "Easy" geometry/topology
- Complicated pieces: polytopes.



dual of $P_{U_{4,2}}$.

$\mathfrak{X}(M)$

- Hard geometry/topology.
- "Easy" pieces.



$\mathfrak{X}(\mathcal{U}_{4,2}) \simeq K_4$.

Idea

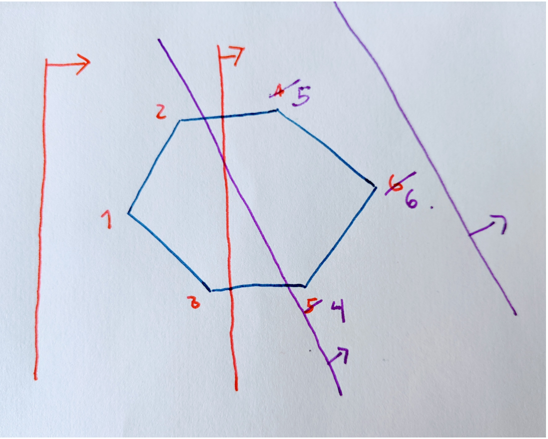
- Line shellings are not enough (translation of Dall).

Idea

- Line shellings are not enough (translation of Dall).
- But we can wiggle the linear functionals along the way and still get shelling orders.

Idea

- Line shellings are not enough (translation of Dall).
- But we can wiggle the linear functionals along the way and still get shelling orders.



Some experimental results of wiggling

With the help of a computer[1].

- The potential family of posets grows substantially!

[1] Find code at: https://github.com/alexheaton2/BrokenLineShellings

Some experimental results of wiggling

With the help of a computer[1].

- The potential family of posets grows substantially!
- The new posets often behave like internal activity: theorems of Las Vergnas/Dawson still hold.

[1] Find code at: https://github.com/alexheaton2/BrokenLineShellings

Some experimental results of wiggling

With the help of a computer[1].

- The potential family of posets grows substantially!
- The new posets often behave like internal activity: theorems of Las Vergnas/Dawson still hold.
- We need to focus on special kinds of wiggles!
- We can find many posets witnessing Stanley's conjecture. They seem to perform the tiny fixes one encounters when starting with an order.

[1] Find code at: https://github.com/alexheaton2/BrokenLineShellings

Some experimental results of wiggling

With the help of a computer[1].

- The potential family of posets grows substantially!
- The new posets often behave like internal activity: theorems of Las Vergnas/Dawson still hold.
- We need to focus on special kinds of wiggles!
- We can find many posets witnessing Stanley's conjecture. They seem to perform the tiny fixes one encounters when starting with an order.
- (Actually proved) The resulting posets result from cutting and pasting several internal activity posets.

[1] Find code at: https://github.com/alexheaton2/BrokenLineShellings

Some new goals

- Are there nice wiggles that preserve nice properties of the posets? (flip theory)

Some new goals

- Are there nice wiggles that preserve nice properties of the posets? (flip theory)
- Can we recover some of the know proofs of special cases using wiggles? Schubert matroids/positroids.

Some new goals

- Are there nice wiggles that preserve nice properties of the posets? (flip theory)
- Can we recover some of the know proofs of special cases using wiggles? Schubert matroids/positroids.
- Internal activity is half of the Tutte Polynomial, can we use the wiggles to study the whole Tutte polynomial too?

Some new goals

- Are there nice wiggles that preserve nice properties of the posets? (flip theory)
- Can we recover some of the know proofs of special cases using wiggles? Schubert matroids/positroids.
- Internal activity is half of the Tutte Polynomial, can we use the wiggles to study the whole Tutte polynomial too?
- Simon's conjecture: Uniform matroid extendably shellable. Dual hypersimplex extendably shellable?

Thank you for your attention.