A Shuffle Theorem for Paths Under any Line

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jointly with

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"Classical" Shuffle Theorem

\[ \nabla e_k (X; q, t) = \sum_{\lambda} t^{\alpha(\lambda)} q^{\binv(\lambda)} \omega G_{\nu(\lambda)} (X; q^{-1}) \]

Conjectured: Haglund, H., Loehr, Remmel, Ulyanov '05

Proved: Carlsson & Mellit '18

Let's go over the ingredients...
\[ \nabla e_k(X; q, t) = \sum_{\lambda} t^{|\lambda|} q^{\text{dim} \nu(\lambda)} \omega \ G_{\nu(\lambda)}(X; q^{-1}) \]

→ \nabla is a symmetric function operator defined in terms of Macdonald polynomials:

\[ \nabla \tilde{H}_\mu = t^{\pi(\mu)} q^{\pi(\mu')} \tilde{H}_\mu \]

→ \nabla e_k gives the doubly graded character of diagonal coinvariants for \( S_k \) (using Hilbert scheme, ...)

\[ \text{linked by Shuffle theorem} \]

Combinatorics of Dyck paths, parking functions ...
\[ \nabla e_{\lambda}(X; q, t) = \sum_{\lambda} t^{a(\lambda)} q^{\text{dim}(\lambda)} \omega G_{\nu, \alpha}(X; q^{-1}) \]

\( \lambda \) is a Dyck path

= lattice path under this line

\( a(\lambda) = \text{area between } \lambda \text{ and the highest path} \)

\( a(\lambda) = 18 \)
\[ \nabla v_\lambda(X; q, t) = \sum_{\lambda} t^{a(\lambda)} q^{\dimv(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1}) \]

\[ \dimv(\lambda) = \# \text{ of balanced hooks in the Young diagram bounded by } \lambda. \]

\[ \frac{l}{a+1} < 1 - \varepsilon < \frac{l+1}{a} \]

\[ a = l + 1 \text{ or } a = l \]
\[ \nabla \mathbf{e}_k(X; q, t) = \sum_{\lambda} t^{\alpha(\lambda)} q^{d_{\text{inv}}(\lambda)} \omega \ G_{\nu(\lambda)}(X; q^{-1}) \]

\[ \rightarrow G_{\nu(\lambda)}(X; q) \] is an LLT polynomial for a tuple of one-row shapes \( \nu(\lambda) \)

box \( j \) in \( \nu^{(i)} \)

\[ j + \xi i \] (same order) \[ -(y + p \times) \]

\[ p = 1 - \xi \]
LLT polynomials

\[ G_v(x_i, q) = \sum_{T \in SSYT(v)} q^{i(T)} x^T \]

\[ i(T) = \# of \text{ attacking inversions:} \]
between entries in boxes such that \( 0 < (j' + e_i') - (j + e_i) \leq 1 \)

LLT's are symmetric (elementary but not obvious) and Schur positive (by K-L theory).
Example

\( k = 3 \)

Diagram with labeled axes and equations:

- \( q^0 \)
- \( q^1 \)
- \( q^2 \)
- \( q^3 \)

- \( \nabla e_3 \)

- \( t^3 \)
- \( q^3 \)
- \( q^t \)
- \( q^{t^2} \)
- \( t^a(x) \)
- \( q \cdot \sin(x) \)
(km, kn) Extended Shuffle Theorem

Conjectured: F. Bergeron, Garsia, Sergel Leven, Xin ’16
Proved: Mellit ’16

(km, kn) = positive integers, expressed with m, n coprime

e_k \left[ -M X^m, n \right] \cdot 1
= \sum_{\lambda} t^{\alpha_{\lambda}} \dim \nu(\lambda) \omega \Gamma_{\nu(\lambda)}(X; q^{-1})

\rho = \frac{n}{m} - \varepsilon

(0, kn)

(km, 0)
\[ e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\lambda} q^{|\lambda|-\lambda_1} \omega G_{\omega(\lambda)}(X ; q^{-1}) \]

\[ e_k[-MX^{m,n}] \] is an operator in the Schiffmann algebra \( \mathcal{E} \) such that
\[ e_k[-MX^{m,1}] \cdot 1 = V^m e_k \]

\[ \text{din}_{\nu}(\lambda) = \# \text{ of } \rho \text{-balanced hooks in the Young diagram bounded by } \lambda, \text{ defined by } \]

\[ y + px = c \]
\[ \frac{l}{a+1} < \rho < \frac{l+1}{a} , \text{ where } \rho = \frac{m}{m} - \varepsilon \]
Theorem: Given real numbers $r, s > 0$ such that \((\text{wlog})\  p = s/r\) is irrational,

$$D_{(b_1, \ldots, b_k), \lambda} = \sum_{\lambda} t^{a_{\lambda}(r)} q^{\text{dim}_p(\lambda)} \omega G_m(\lambda)(X; q^{-1}),$$

where $\lambda$ is a lattice path under the line $y + px = s$, $(b_1, \ldots, b_k)$ are the South runs on the highest such path, and $D_{\lambda} \in \mathcal{E}$ is a Feigin-Tsymbaliuk element—which reduces to $e_k [-M X_{m,n}]$ for $(r, s) = (km, kn)$.
Hints on the proof → The LHS is the polynomial part of a raising operator series:

\[ \omega(D_b \cdot 1) = \left( \sum_{w \in S_2} \omega \left( \frac{x^b \prod_{i<j} \epsilon_j (1 - q^b x_i / x_j)}{\prod_{i<j} (1-x_i/x_j)(1-q x_i/x_j)(1+t x_i/x_j)} \right) \right)_{+} \]

→ The LLT pol \( G_2(x; q^{-1}) \) is the polynomial part of an LLT series (Grojnowski & H. ’07) \( \mathcal{L}_{\mathbb{N}}(x; q) \), up to a factor \( q^{\infty} \).

→ We prove a stronger identity of formal power series

\[ \Phi_D(b_1, \ldots, b_k) = \sum_{a_1, \ldots, a_{k-1} \geq 0} t^{a_1} \mathcal{L}(b_1, \ldots, b_k) \mathcal{L}(0, a_1, \ldots, a_{k-1});(0, a_k, \ldots, a_{k-1}, 0)(x; q), \]

where \( \Phi_D(b_1, \ldots, b_k) \) is the full raising operator series above.
\[ \Phi D(b_1, \ldots, b_k) = \sum_{a_1} \frac{\lambda^{a_1}}{(a_1 + (0, a_2, \ldots, a_n))} / (a_1, \ldots, a_n, 0)(x, q) \]

→ In the polynomial part, one term in the sum survives for each path \( \lambda \) under the line, with \( |a_1| = a(\lambda) \).

→ In the term for \( \lambda \), it turns out that
\[ \lambda a(n; q) \text{ pole } = q^{\text{deg}(\lambda)} G_{\text{uni}}(x, q^{-1}). \]

→ The series formulation is essentially a corollary to a Cauchy formula for nonsymmetric Hall-Littlewood polynomials.
What else?

- We don't know whether a 'compositional' shuffle theorem is possible for paths under an arbitrary line.

We can prove some other conjectures with this method—writing in progress:

- The 'extended Delta conjecture' of Haglund, Remmel, Wilson '18 for $\Delta_{\Lambda} \Delta'_{e_m e_k}$

- The conjecture of Loehr-Warrington '08 for $\nabla s_{\lambda}$