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A Shuffle Theorem for Paths Under any Line

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jointly with

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'Classical' Shuffle Theorem

$$\nabla e_k(X;q,t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X;q^{-1})$$

Conjectured: Haglund, H., Loehr, Remmel, Ulyanov '05

Proved: Carlsson + Mellit '18

Let's go over the ingredients ...

$$\nabla e_k(X;q,t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}(\lambda)} \, \omega \, G_{\nu(\lambda)}(X;q^{-1})$$

→ ∇ is a symmetric function operator defined in terms of Macdonald polynomials:

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu$$

→ ∇e_k gives the doubly graded character of diagonal coinvariants for S_k (using Hibert scheme, ...)

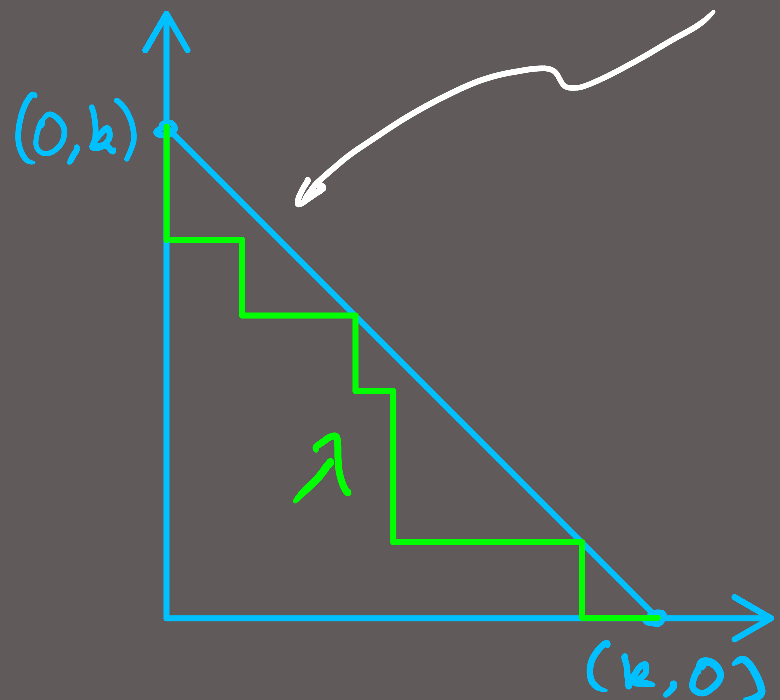
linked by Shuffle theorem

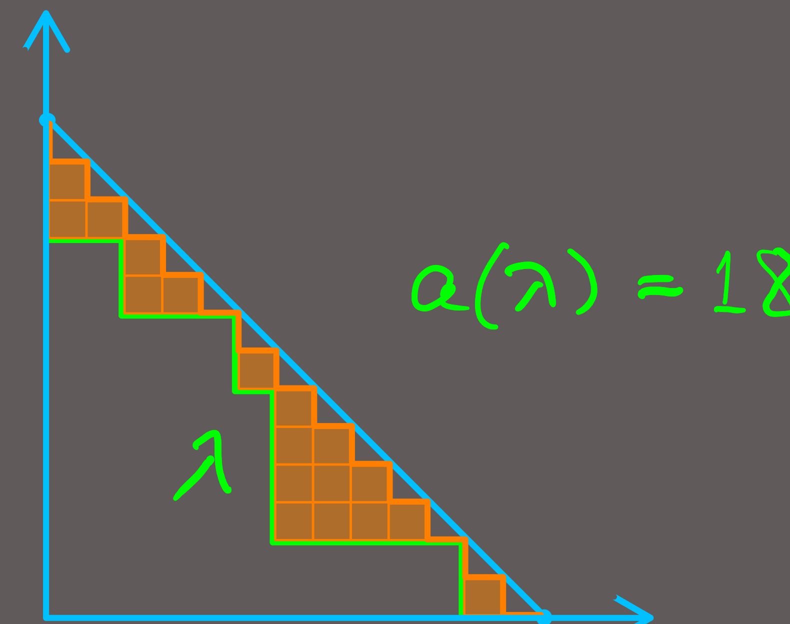
Combinatorics of Dyck paths, parking functions...

$$\nabla e_k(X;q,t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X;q^{-1})$$

λ is a Dyck path
= lattice path under this line

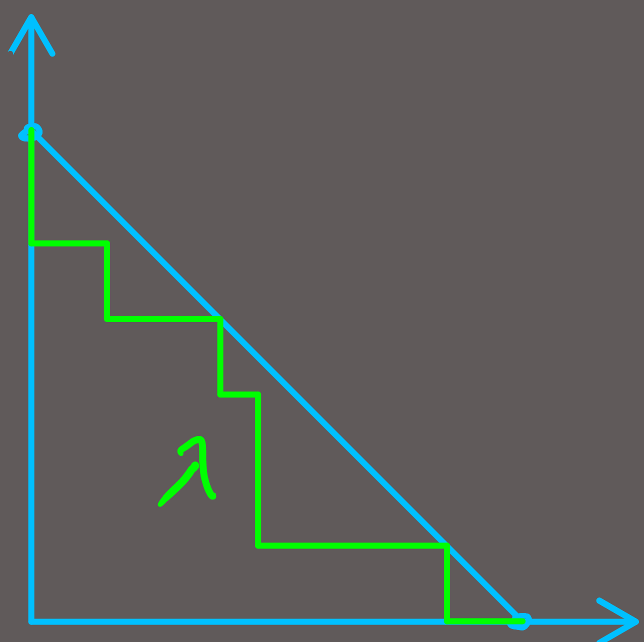
$a(\lambda)$ = area between λ and the highest path

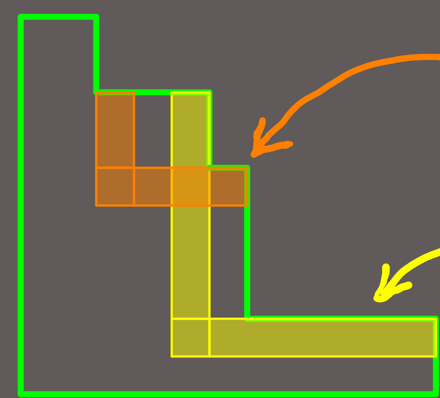




$$\nabla e_k(X;q,t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}(\lambda)}\, \omega\, G_{\nu(\lambda)}(X;q^{-1})$$

→ $\mathrm{dinv}(\lambda)$ = # of **balanced hooks** in the Young diagram bounded by λ.





$a = \ell + 1$

or

$a = \ell$

$\Updownarrow$

$$\frac{\ell}{a+1} < 1-\varepsilon < \frac{\ell+1}{a}$$

$$\nabla e_k(X;q,t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}(\lambda)}\, \omega\, G_{\nu(\lambda)}(X;q^{-1})$$

$\rightarrow$ $G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of one-row shapes $\nu(\lambda)$

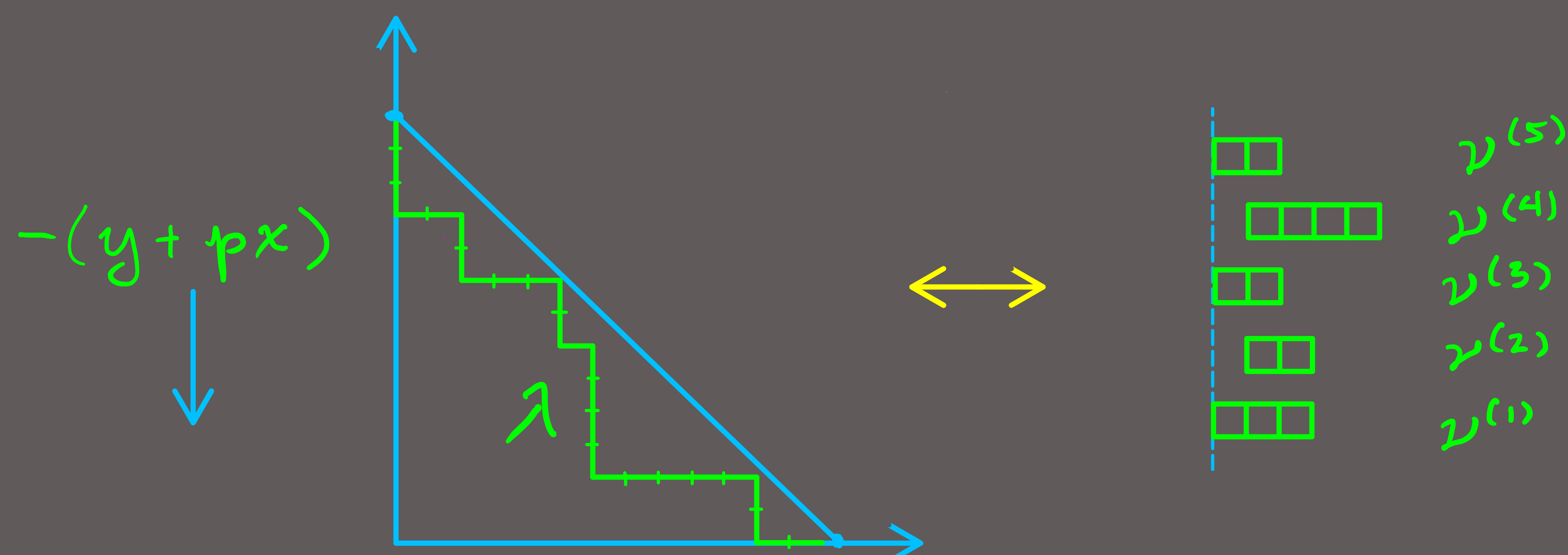
box j in $\nu^{(i)}$

$\nu^{(i)} \longleftrightarrow$ South runs in λ

$j+\varepsilon i \longleftrightarrow -(y+px)$

(same order)

$p = 1-\varepsilon$

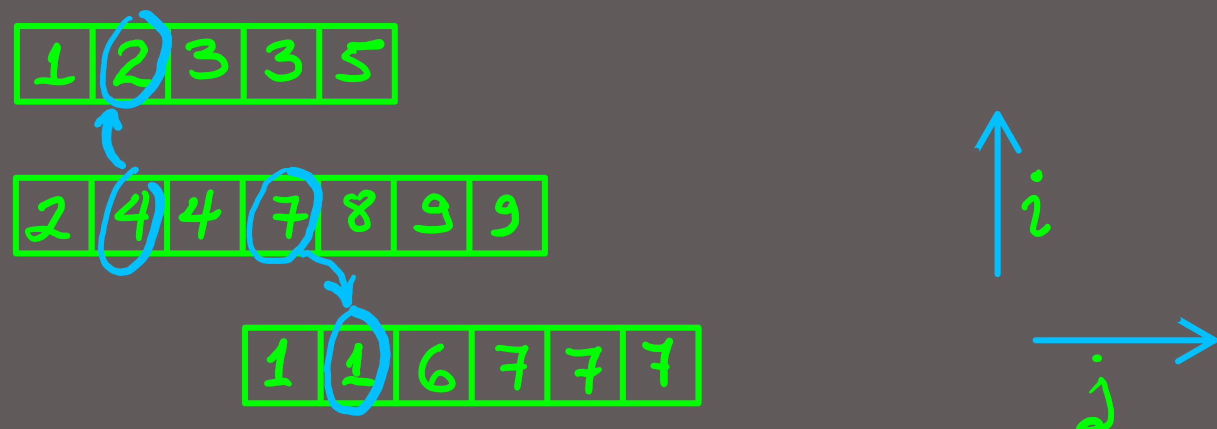


LLT polynomials

$$G_\nu(X;q) = \sum_{T \in SSYT(\nu)} q^{i(T)} x^T$$

$i(T)$ = # of attacking inversions:

between entries in boxes such that $0 < (j'+\varepsilon i') - (j+\varepsilon i) < 1$



LLT's are symmetric (elementary but not obvious)

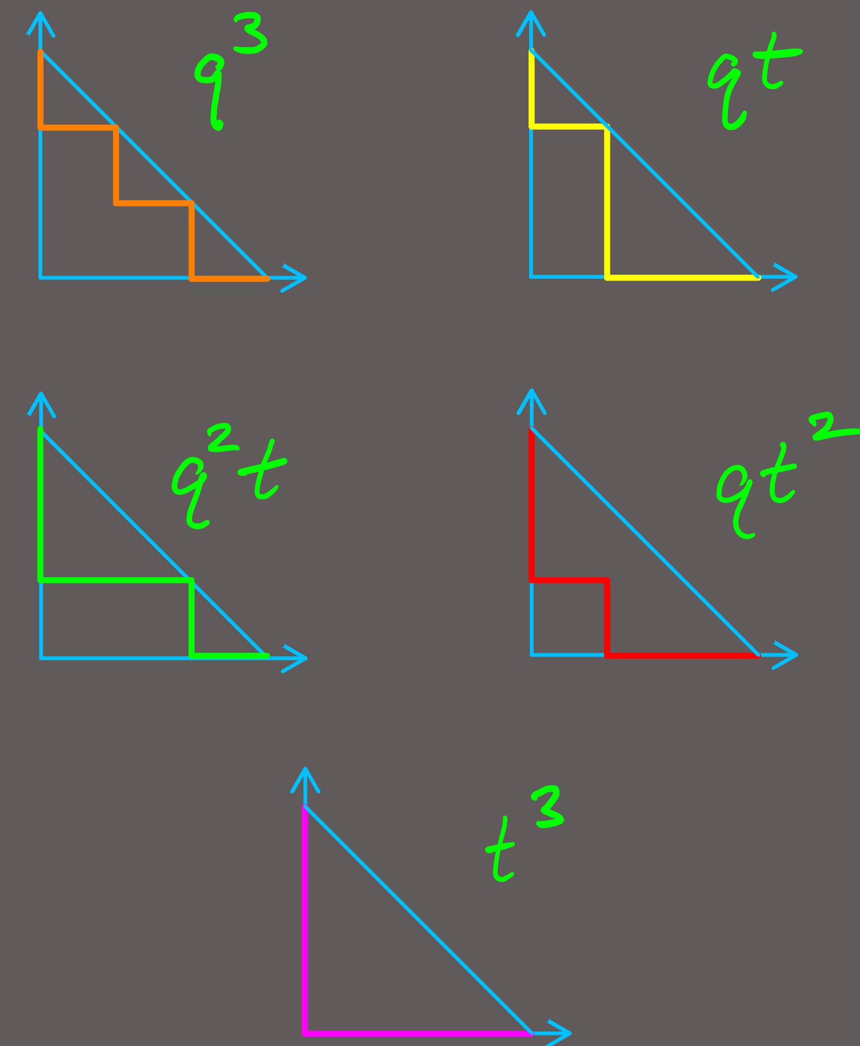
and Schur positive (by K-L theory).

Example

$k = 3$

	q^0	q^1	q^2	q^3
t^3	s_{111}			
t^2	s_{21}	s_{111}		
t^1	s_{21}	$s_{111} + s_{21}$	s_{111}	
t^0	s_{3}	s_{21}	s_{21}	s_{111}

∇e_3



$t^{a(\lambda)} \, q^{\mathrm{dinv}(\lambda)}$

(km, kn) Extended Shuffle Theorem

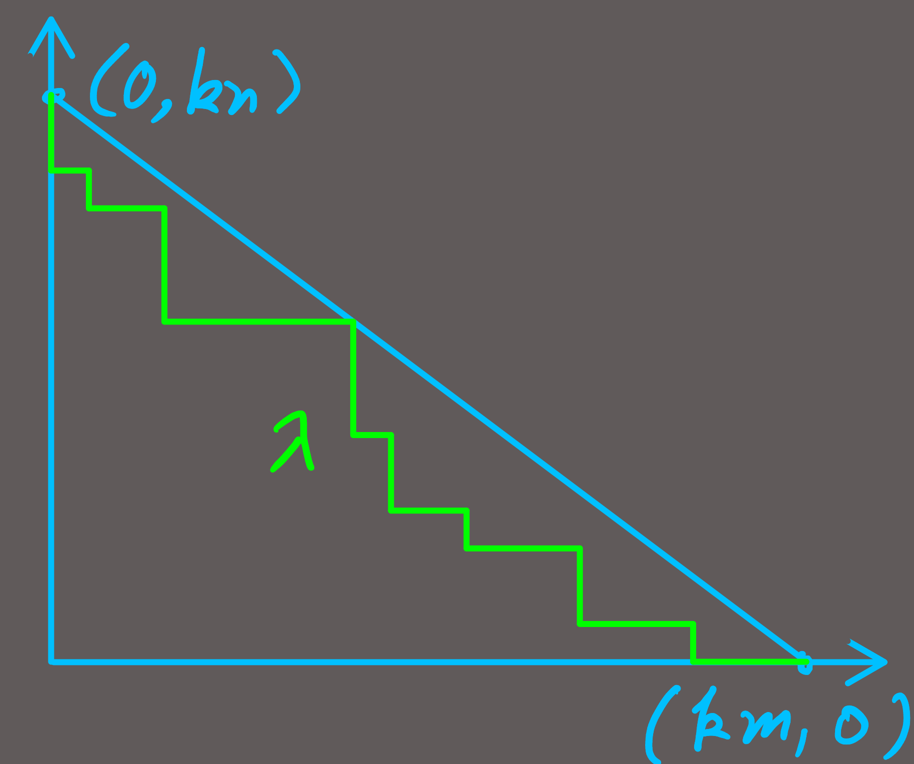
Conjectured: F. Bergeron, Garsia, Sergel Leven, Xin '16

Proved: Mellit '16

(km, kn) = positive integers, expressed with m, n coprime

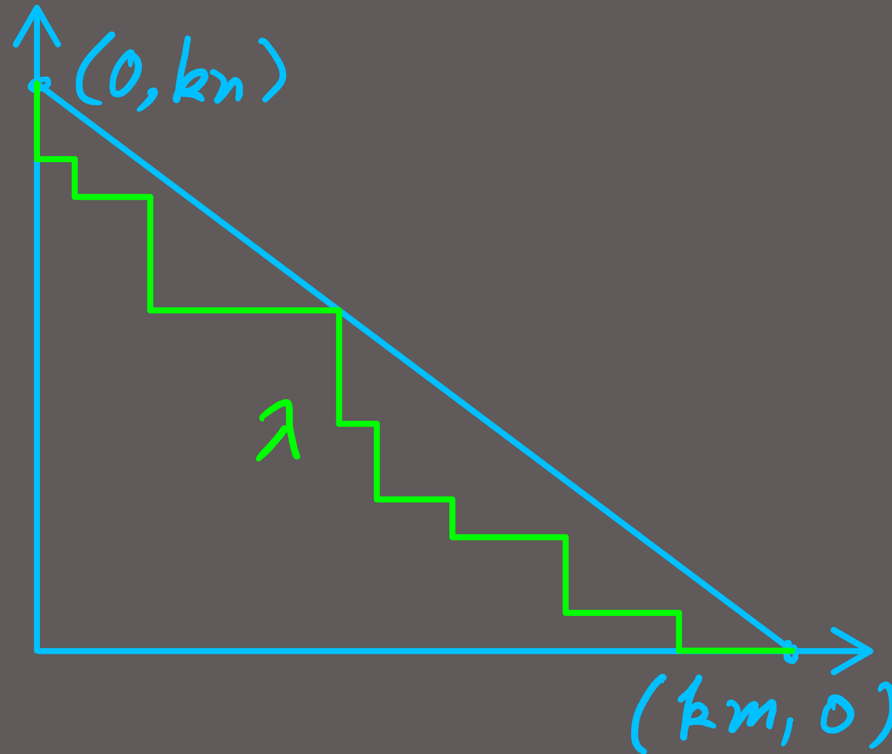
$$e_k[-MX^{m,n}]\cdot 1 = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)}\, \omega\, G_{\nu(\lambda)}(X; q^{-1})$$

$$p = \frac{n}{m} - \varepsilon$$



$$e_k[-MX^{m,n}]\cdot 1$$

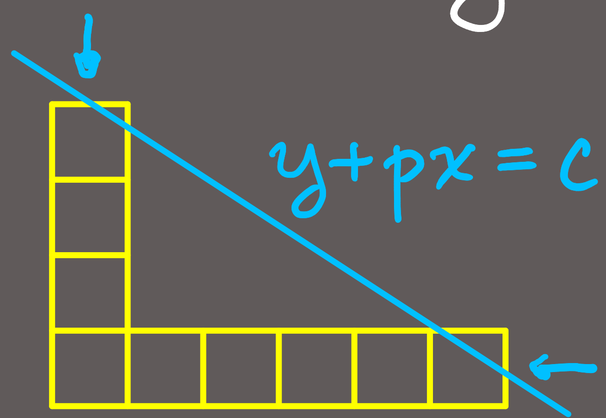
$$= \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)}\, \omega G_{\nu(\lambda)}(X; q^{-1})$$



→ $e_k[-MX^{m,n}]$ is an operator in the Schiffmann algebra $\mathcal{E}$ such that

$$e_k[-MX^{m,1}]\cdot 1 = \nabla^m e_k$$

→ $\mathrm{dinv}_p(\lambda)$ = # of p-balanced hooks in the Young diagram bounded by λ, defined by

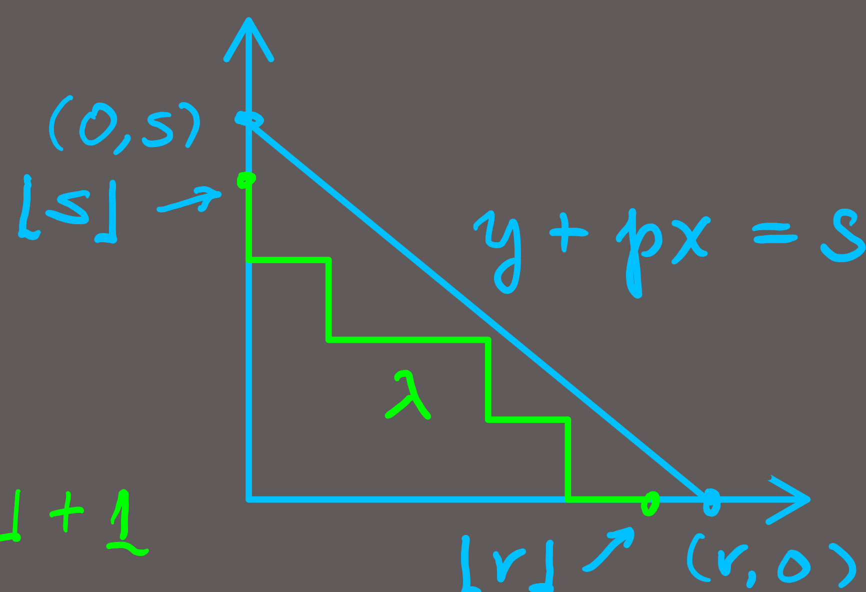


$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a}, \text{ where } p = \frac{n}{m} - \varepsilon$$

Theorem Given real numbers $r, s > 0$ such that (wlog) $p = s/r$ is irrational,

$$D_{(b_1,\dots,b_\ell)}\cdot 1 = \sum_\lambda t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)}\, \omega\, C_{\mathrm{rv}(\lambda)}(X; q^{-1}),$$

where λ is a lattice path under the line $y + px = s$, $(b_1,\dots,b_\ell)$ are the South runs on the highest such path, and $D_{\underline{b}} \in \mathcal{E}$ is a Feigin-Tsymbauliak element — which reduces to $e_k[-M X^{m,n}]$ for $(r,s) = (km, kn)$.



$\ell = \lfloor r \rfloor + 1$

Hints on the proof → The LHS is the polynomial part of a raising operator series:

$$\omega(D_{\underline{b}} \cdot 1) = \left(\sum_{w \in S_\ell} w \left(\frac{x^{\underline{b}} \prod_{i+1<j} (1 - qt\, x_i/x_j)}{\prod_{i<j} \left((1 - x_j/x_i)(1 - q x_i/x_j)(1 - t x_i/x_j) \right)} \right) \right)_{\mathrm{pol}}$$

→ The LLT pol $G_\nu(X; q^{-1})$ is the polynomial part of an LLT series (Grojnowski + H. '07) $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)$, up to a factor $q^{(-)}$.

→ We prove a stronger identity of formal power series

$$\Phi D_{(b_1,\dots,b_\ell)} = \sum_{a_1,\dots,a_{\ell-1} \geq 0} t^{|\underline{a}|} \mathcal{L}^{\sigma}_{((b_\ell,\dots,b_1)+(0,a_{\ell-1},\dots,a_1))/(a_{\ell-1},\dots,a_1,0)}(x;q),$$

where $\Phi D_{(b_1,\dots,b_\ell)}$ is the full raising operator series above.

$$\Phi D_{(b_1,\dots,b_\ell)} = \sum_{a_1,\dots,a_{\ell-1}\geq 0} t^{|\underline{a}|}\, \mathcal{L}^{\sigma}_{((b_\ell,\dots,b_1)+(0,a_{\ell-1},\dots,a_1))/(a_{\ell-1},\dots,a_1,0)}(x;q)$$

→ In the polynomial part, one term in the sum survives for each path λ under the line, with $|\underline{a}| = a(\lambda)$.

→ In the term for λ, it turns out that

$$\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)_{\mathrm{pol}} = q^{\dim_p(\lambda)}\, G_{\nu(\lambda)}(x;q^{-1}).$$

→ The series formulation is essentially a corallary to a Cauchy formula for nonsymmetric Hall-Littlewood polynomials.

What else?

- We don't know whether a 'compositional' shuffle theorem is possible for paths under an arbitrary line.

We can prove some other conjectures with this method — writing in progress:

- The 'extended Delta conjecture' of Haglund, Remmel, Wilson '18 for $\Delta_{h_\ell}\Delta'_{e_m}e_k$

- The conjecture of Loehr-Warrington '08 for ∇s_λ