Representations of Multiplier Algebras in Spaces of Completely Bounded Maps

Nico Spronk

ABSTRACT. If G is a locally compact group, then the measure algebra $\mathrm{M}(G)$ and the completely bounded multipliers of the Fourier algebra $\mathrm{M}_{cb}\mathrm{A}(G)$ can be seen to be dual objects to one another in a sense which generalises Pontryagin duality for abelian groups. We explore this duality in terms of representations of these algebras in spaces of completely bounded maps.

This article is intended to give a tour of the growing body of work on representing multiplier algebras in spaces of maps on the C*-algebra $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The ideas here begin in the 1980s with the work of Størmer [**37**], on representations of measure algebras of abelian groups; work of Ghahramani [**12**], on a representation of group algebras for general groups; and unpublished work of Haagerup [**13**], on a representation of completely bounded multipliers of the Fourier algebra of a general locally compact group. Størmer's and Ghahramani's results have been expanded upon and improved in the work of Neufang [**22**], Neufang, Ruan and Spronk [**23**] and Smith and Spronk [**33**]. The results of Haagerup were rediscovered by Spronk [**34, 35**]. These ideas are currently being reinterpreted by Neufang, Ruan and Spronk, and may yield results on multipliers of "quantum groups".

Date: March 2, 2005.

2000 *Mathematics Subject Classification.* Primary 46L07, 22D20, 43A20; Secondary 22D10,22D25. *Key words and phrases.* Group algebra, Fourier algebra, completely bounded map, extended Haagerup tensor product.

The author was supported by an NSERC PDF.

1. Harmonic Analysis

For any Banach space $\mathcal{X}$ we let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded linear operators on $\mathcal{X}$.

Let G be a locally compact group. Let, for $1 \leq p \leq \infty$, $\mathrm{L}^p(G)$ denote the space of (almost everywhere equivalence classes) of functions on G which are p-integrable with respect to integration over the left Haar measure $\int_G \cdots ds$. We will let $\mathrm{M}(G)$ denote the usual *measure algebra* over G, which is the dual of the space of continuous functions vanishing at infinity $\mathcal{C}_0(G)$. We note that $\mathrm{M}(G)$ also injects into the dual of the space of bounded continuous functions $\mathcal{C}_b(G)$. All of these spaces are discussed in the standard references [**16, 24, 30**].

Let $\mathcal{H}$ be a Hilbert space and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation. Then there exists a homomorphism $\pi_1 : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H})$ given for each μ in $\mathrm{M}(G)$ by the vector integral $\int_G \pi(s)d\mu(s)$, where the integral may be considered to converge in the weak* topology in $\mathcal{B}(\mathcal{H})$. We identify the *group algebra* $\mathrm{L}^1(G)$ with the ideal of measures in $\mathrm{M}(G)$ which are absolutely continuous with respect to Haar measure. We identify the *discrete group algebra* $\ell^1(G)$ with the closed span of the Dirac measures on G. We then let

$$\mathrm{C}^*_\pi = \overline{\pi_1(\mathrm{L}^1(G))}, \quad \mathrm{D}^*_\pi = \overline{\pi_1(\ell^1(G))} \quad \text{and} \quad \mathrm{M}^*_\pi = \overline{\pi_1(\mathrm{M}(G))}$$

where closures are each taken in the norm topology of $\mathcal{B}(\mathcal{H})$. We note that M^*_π is contained in MC^*_π, the multiplier C*-algebra of C^*_π. However, $\mathrm{M}^*_\pi \neq \mathrm{MC}^*_\pi$, in general; an example may be found in taking $G = \mathbb{R}$ and π to be the left regular representation.

The spaces $\mathrm{A}(G)$ and $\mathrm{B}(G)$ are the *Fourier* and *Fourier-Stieltjes algebras*, as introduced in Eymard [**11**]. If G is abelian with Pontryagin dual group $\widehat{G}$, then the Fourier transform identifies $\mathrm{A}(G) \cong \mathrm{L}^1(\widehat{G})$ and $\mathrm{B}(G) \cong \mathrm{M}(\widehat{G})$. Thus $\mathrm{A}(G)$ and $\mathrm{B}(G)$ play roles analogous to $\mathrm{L}^1(G)$ and $\mathrm{M}(G)$. The multipliers of $\mathrm{A}(G)$ are given by

$$\mathrm{MA}(G) = \{u : G \to \mathbb{C} | uv \in \mathrm{A}(G) \text{ for each } v \in \mathrm{A}(G)\}.$$

An application of the Closed Graph Theorem shows that each multiplier automatically gives rise to a bounded linear operator $m_u : \mathrm{A}(G) \to \mathrm{A}(G)$, $m_u v = uv$.

Thus $\mathrm{MA}(G)$ is a commutative Banach subalgebra of $\mathcal{B}(\mathrm{A}(G))$. The Fourier algebra $\mathrm{A}(G)$ is the predual of the group von Neumann algebra $\mathrm{VN}(G)$, and as such obtains a natural structure as an *operator space*, as has been observed by Blecher [**2**] and by Effros and Ruan [**8**]. It thus makes sense to define the *completely bounded multipliers*

$$\mathrm{M}_{cb}\mathrm{A}(G) = \{u \in \mathrm{MA}(G) | m_u \in \mathcal{CB}(\mathrm{A}(G))\}$$

where $\mathcal{CB}(\mathrm{A}(G))$ is the algebra of *completely bounded* linear maps on $\mathrm{A}(G)$. (For a full exposition of operator spaces and completely bounded maps, we refer the reader to [**9**].) The completely bounded multipliers first studied, in a dual context as maps on $\mathrm{VN}(G)$, by de Cannière and Haagerup [**7**] and further studied in [**6, 5**].

Wendel's Theorem [**41**] tells us that $\mathrm{M}(G)$ is the multiplier algebra of $\mathrm{L}^1(G)$. Thus if G is abelian, we see that $\mathrm{MA}(G) = \mathrm{B}(G)$. In fact, we always have that $\mathrm{B}(G) \subset \mathrm{M}_{cb}\mathrm{A}(G) \subset \mathrm{MA}(G)$. Losert [**21**] has shown that $\mathrm{B}(G) = \mathrm{MA}(G)$ exactly when G is amenable (see [**25, 30**] for more on amenable groups). It is also known, due to an unpublished result of Losert, that $\mathrm{B}(G) = \mathrm{M}_{cb}\mathrm{A}(G)$ exactly when G is amenable; this was also proved by Bożejko [**4**] in the case that G is discrete. It is unknown if the equality $\mathrm{M}_{cb}\mathrm{A}(G) = \mathrm{MA}(G)$ characterises the amenability of G.

Hence we see that each of $\mathrm{B}(G)$, $\mathrm{M}_{cb}\mathrm{A}(G)$ and $\mathrm{MA}(G)$ is a reasonable candidate for the "dual object" of $\mathrm{M}(G)$ in the sense that $\mathrm{A}(G)$ is the dual object for $\mathrm{L}^1(G)$. In this article we aim to establish a context in which $\mathrm{M}_{cb}\mathrm{A}(G)$ is the most reasonable dual object. Another context, one which uses similar techniques to ours, is worked out by Walter [**39, 40**]. In that work, $\mathrm{B}(G)$ is the appropriate dual object. The relationship between Walter's results and ours seems worthy of deeper study.

Let us finish with a brief note about operator space structures. Though any normed linear space may be endowed with a multitude of operator space structures, there are certain canonical choices, developed in [**9, 2**], which are most natural for certain types of normed linear spaces. We have that $\mathrm{B}(G)$ is the dual of the enveloping C*-algebra $\mathrm{C}^*(G)$, and hence we will always assign it the standard dual operator space structure. As mentioned above, $\mathrm{A}(G)$ obtains the standard predual operator space structure as the predual of $\mathrm{VN}(G)$. However, it may be shown that this is the same as the operator subspace structure it inherits being a subspace of $\mathrm{B}(G)$. If $\mathcal{V}$ is any operator space, the algebra of completely bounded linear maps on it, $\mathcal{CB}(\mathcal{V})$, will be given the standard operator space structure. Any subspace of $\mathcal{CB}(\mathcal{V})$, such as $\mathrm{M}_{cb}\mathrm{A}(G)$ which imbeds into $\mathcal{CB}(\mathrm{A}(G))$, will inherit the operator subspace structure. Any C*-algebra will have its C*-algebra operator space structure. In particular, any commutative C*-algebra will be a minimal operator space. Any space $\mathrm{L}^1(X,\mu)$, for a measure space (X,μ), will have the maximal operator space structure. If X is further a locally compact topological space we assign its measure space $\mathrm{M}(X)$ the maximal operator space structure. The maximal operator space structure is the standard dual (resp. predual) structure in terms of the dual pairing $\mathcal{C}_0(X)^* \cong \mathrm{M}(X)$ (resp. $\mathrm{L}^\infty(X,\mu)_* \cong \mathrm{L}^1(X,\mu)$).

2. Spaces of Completely Bounded Normal Maps

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ denote the space of all normal completely bounded maps on $\mathcal{B}(\mathcal{H})$. The following can be obtained from the representation theorem for completely bounded maps (see [**13, 26, 9**]) applied to the algebra $\mathcal{K}(\mathcal{H})$, of compact operators on $\mathcal{H}$.

Theorem 2.1. [**13, 32**] *$T \in \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$ if and only if there exist families $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ from $\mathcal{B}(\mathcal{H})$, where the cardinality of the index set I and the Hilbertian dimension of $\mathcal{H}$ are related by $|I| = (\dim\mathcal{H})^2$, such that for any x in $\mathcal{B}(\mathcal{H})$*

$$Tx = \sum_{i\in I} a_i x b_i \quad \text{and} \quad \|T\|_{cb} = \left\|\sum_{i\in I} a_i a_i^*\right\|^{1/2} \left\|\sum_{i\in I} b_i^* b_i\right\|^{1/2}$$

where all series converge in the weak topology of $\mathcal{B}(\mathcal{H})$.*

We thus define the *extended Haagerup tensor product* $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ by identifying it with $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$. Hence elements of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ are (formal) series of elementary tensors $T = \sum_{i\in I} a_i \otimes b_i$ where $\|T\|_{eh}^2 = \left\|\sum_{i\in I} a_i a_i^*\right\| \left\|\sum_{i\in I} b_i^* b_i\right\| < \infty$. We identify series $\sum_{i\in I} a_i \otimes b_i$ and $\sum_{i\in I} a_i' \otimes b_i'$ provided $\sum_{i\in I} a_i x b_i = \sum_{i\in I} a_i' x b_i'$ for each x in $\mathcal{B}(\mathcal{H})$. $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ is a Banach algebra when the multiplication is induced by its identification with $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$, so on elementary tensors we have multiplication $(a \otimes b)\circ(c \otimes d) = ac \otimes db$. In this context, we note that the Haagerup tensor product $\mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$ is the $\|\cdot\|_{cb}$-closure of the space of elementary operators on $\mathcal{B}(\mathcal{H})$. Moreover, each T in $\mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$ can be represented $T = \sum_{i=1}^\infty a_i \otimes b_i$ where the series $\sum_{i=1}^\infty a_i a_i^*$ and $\sum_{i=1}^\infty b_i^* b_i$ converge uniformly. See [**32**] for this description of $\mathcal{B}(\mathcal{H}) \otimes^h \mathcal{B}(\mathcal{H})$.

If $F \in \mathcal{B}(\mathcal{H})^*$ we define the *left* and *right slice* maps $L_F, R_F : \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$L_F T = \sum_{i \in I} F(a_i) b_i \quad \text{and} \quad R_F T = \sum_{i \in I} F(b_i) a_i$$

where $T = \sum_{i \in I} a_i \otimes b_i$. That L_F and R_F are well defined operators and that these series converge in norm is shown in [**35**]. The following is from [**34, 35**], which is adapted from [**3, 32**].

Theorem 2.2. *If $\mathcal{V}$ and $\mathcal{W}$ are closed subspaces of $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ then the following are equivalent:*

(i) *$L_F T \in \mathcal{W}$ and $R_F T \in \mathcal{V}$ for each F in $\mathcal{B}(\mathcal{H})^*$.*

(ii) *$L_\omega T \in \mathcal{W}$ and $R_\omega T \in \mathcal{V}$ for each ω in $\mathcal{B}(\mathcal{H})_*$.*

(iii) *there exist families $\{v_i\}_{i \in I}$ from $\mathcal{V}$ and $\{w_i\}_{i \in I}$ from $\mathcal{W}$ such that $T = \sum_{i \in I} v_i \otimes w_i$ and $\|T\|_{eh} = \left\|\sum_{i \in I} v_i v_i^*\right\|^{1/2} \left\|\sum_{i \in I} w_i^* w_i\right\|^{1/2}$.*

The space of all T satisfying any of the above conditions is thus denoted $\mathcal{V} \otimes^{eh} \mathcal{W}$. The spaces $\mathcal{V} \otimes^{eh} \mathcal{W}$, as well as extended Haagerup tensor products of multiple operator spaces, are also developed by Effros and Ruan [**10**], using different methods. Their goal in that article is to study "operator convolution algebras", a goal similar to ours in the present article.

Let us summarise some properties of the extended Haagerup tensor product.

Theorem 2.3. (i) [**3, 10**] *If $\mathcal{V}$ and $\mathcal{W}$ are dual operator spaces with respective preduals $\mathcal{V}_*$ and $\mathcal{W}_*$, then $\mathcal{V} \otimes^{eh} \mathcal{W} \cong (\mathcal{V}_* \otimes^h \mathcal{W}_*)^*$.*

(ii) [**33, 1**] *If $\mathcal{A}$ and $\mathcal{B}$ are each subalgebras of $\mathcal{B}(\mathcal{H})$ then $\mathcal{A} \otimes^{eh} \mathcal{B}$ is a subalgebra of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$. If $\mathcal{I}$ is a left ideal of $\mathcal{A}$ and $\mathcal{J}$ is a right ideal of $\mathcal{B}$, then $\mathcal{I} \otimes^{eh} \mathcal{J}$ is a left ideal of $\mathcal{A} \otimes^{eh} \mathcal{B}$.*

(iii) [**32, 3, 13**] *If $\mathcal{M}$ is a locally cyclic C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, i.e. for every finite dimensional subspace $\mathcal{L}$ of $\mathcal{H}$ there is $\xi \in \mathcal{H}$ such that $\overline{\mathcal{M}\xi} \supset \mathcal{L}$, then each $\mathcal{M}$-bimodule map on $\mathcal{B}(\mathcal{H})$ is automatically completely bounded. Moreover, there is a natural identification between the space of normal $\mathcal{M}$-bimodule maps $\mathcal{B}^\sigma_{\mathcal{M}}(\mathcal{B}(\mathcal{H}))$ and $\mathcal{M}' \otimes^{eh} \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$.*

Let us expose these ideas by introducing a class of commutative Banach algebras which has been studied in [**34, 35, 31**]. If X is a locally compact Hausdorff space, then, by Grothendieck's inequality, the algebra $\mathrm{V}_0(X) = \mathcal{C}_0(X) \otimes^h \mathcal{C}_0(X)$ is exactly the algebra $\mathcal{C}_0(X) \otimes^\gamma \mathcal{C}_0(X)$ (projective tensor product) studied by Varopoulos [**38**]. We thus dub it the *Varopoulos algebra*. The *extended Varopoulos algebras* are given by

$$\mathrm{V}^0(X) = \mathcal{C}_0(X) \otimes^{eh} \mathcal{C}_0(X) \quad \text{and} \quad \mathrm{V}^b(X) = \mathcal{C}_b(X) \otimes^{eh} \mathcal{C}_b(X).$$

An element w of $\mathrm{V}^b(X)$ can be written $w = \sum_{i \in I} \varphi_i \otimes \psi_i$, where we have that $\left\| \sum_{i \in I} |\varphi_i|^2 \right\|_\infty \left\| \sum_{i \in I} |\psi_i|^2 \right\|_\infty < \infty$. These series can be taken to converge pointwise in $\ell^\infty(X)$. Thus we may consider w to be a function on $X \times X$ by letting $w(x, y) = \sum_{i \in I} \varphi_i(x)\psi_i(y)$ for each (x, y). As pointed out in [**31**], w need not be continuous on $X \times X$. However, for any fixed x in X, pointwise slices $w(\cdot, x)$ and $w(x, \cdot)$ are in $\mathcal{C}_b(X)$ (and in $\mathcal{C}_0(X)$ if $w \in \mathrm{V}^0(X)$). If (X, μ) is any measure space, we let $\mathrm{V}^\infty(X, \mu) = \mathrm{L}^\infty(X, \mu) \otimes^{eh} \mathrm{L}^\infty(X, \mu)$. Similarly as above, we may consider elements of $\mathrm{V}^\infty(X, \mu)$ to be functions on $X \times X$, but defined only up to locally marginally null sets. We note that if (X, μ) is further assumed to satisfy the Radon-Nikodym Theorem, so $\mathrm{L}^1(X, \mu)^* \cong \mathrm{L}^\infty(X, \mu)$, then $\mathrm{V}^\infty(X, \mu)$ is naturally isomorphic to each of

$$\left(\mathrm{L}^1(X, \mu) \otimes^h \mathrm{L}^1(X, \mu)\right)^* \quad \text{and} \quad \mathcal{B}^\sigma_{\mathrm{L}^\infty(X,\mu)}(\mathcal{B}(\mathrm{L}^2(X, \mu)))$$

where $\mathrm{L}^\infty(X, \mu)$ is identified with the maximal abelian subalgebra of multiplication operators on $\mathrm{L}^2(X, \mu)$. In particular, if μ is counting measure on X, then $\mathrm{V}^\infty(X, \mu) = \ell^\infty(X) \otimes^{eh} \ell^\infty(X)$ is shown in [**34, 35**] to be the algebra of *Schur multipliers* on $\mathcal{B}(\ell^2(X))$. Variants of this fact are established by Hladnik [**17**] and Pisier [**27**]. This justifies referring to the space $\mathrm{V}^\infty(X, \mu)$, for a measure space (X, μ), as the space of *measurable Schur multipliers*.

3. Representations of $\mathrm{M}(G)$

Let G be a locally compact group, $\mathcal{H}$ be a Hilbert space and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous representation. The representation $\pi_1 : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H})$ may retain some information about the Banach algebra $\mathrm{M}(G)$ or its ideal $\mathrm{L}^1(G)$. However, it is shown by Ghahramani [**12**] that it is impossible for there to exist any isometric homomorphism from $\mathrm{L}^1(G)$ into $\mathcal{B}(\mathcal{H})$. Thus it is intriguing to note that π still can be exploited to give a representation of $\mathrm{M}(G)$ which retains some spatial information.

Let $\gamma_\pi : G \to \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ be given by

$$\gamma_\pi(s) = \pi(s) \otimes \pi(s)^*.$$

By the multiplication on $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H}) \cong \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$, γ_π is a homomorphism. We can view the continuity of γ_π in two ways:

(i) $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ has a weak* topology given by the predual $\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$. γ_π is continuous in this weak* topology.

(ii) For each s in G, $\gamma_\pi(s)(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$. The map $s \mapsto \gamma_\pi(s)|_{\mathcal{K}(\mathcal{H})}$ from G to $\mathcal{CB}(\mathcal{K}(\mathcal{H}))$ is continuous when $\mathcal{CB}(\mathcal{K}(\mathcal{H}))$ has the point-norm topology.

We can use γ_π to define a map $\Gamma_\pi : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ by letting for each μ in $\mathrm{M}(G)$,

$$\Gamma_\pi(\mu) = \int_G \gamma_\pi(s) d\mu(s) = \int_G \pi(s) \otimes \pi(s)^* d\mu(s). \tag{3.1}$$

This vector-valued integral can be interpreted in one of two ways. Using (i), we can view (3.1) as converging in the weak* topology of $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$, i.e. for each $x = \sum_{i=1}^\infty \omega_i \otimes \upsilon_i$ in $\mathcal{B}(\mathcal{H})_* \otimes^h \mathcal{B}(\mathcal{H})_*$ we have

$$\langle \Gamma_\pi \mu, x \rangle = \int_G \langle \pi(s) \otimes \pi(s)^*, x \rangle \, d\mu(s) = \sum_{i=1}^\infty \int_G \langle \pi(s), \omega_i \rangle \langle \pi(s)^*, \upsilon_i \rangle \, d\mu(s).$$

On the other hand, by (ii), we can obtain Bochner integrals pointwise on $\mathcal{K}(\mathcal{H})$, i.e. for each k in $\mathcal{K}(\mathcal{H})$ we let

$$\tilde{\gamma}_\pi(\mu) k = \int_G \pi(s) k \pi(s)^* d\mu(s).$$

Then $\tilde{\gamma}_\pi : \mathrm{M}(G) \to \mathcal{B}(\mathcal{K}(\mathcal{H}))$ is a bounded homomorphism whose range may be verified to be in $\mathcal{CB}(\mathcal{K}(\mathcal{H}))$. We can then view each operator $\Gamma_\pi(\mu) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ as the second Banach space adjoint $\tilde{\gamma}_\pi(\mu)^{**}$. Thus $\Gamma_\pi : \mathrm{M}(G) \to \mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ is a homomorphism. We note that this fact is established in [**33**], using (i) above; the proof is complicated by the fact that the Banach algebra $\mathcal{B}(\mathcal{H}) \otimes^{eh} \mathcal{B}(\mathcal{H})$ does not have weak* continuous multiplication on the right, in general.

We now consider the left regular representation λ. The next result is due to Størmer [**37**] in the case that G is abelian, and due to Gharamani [**12**] in general. It is recognised by Neufang [**22**] that the image is one of completely bounded maps.

THEOREM 3.1. *The map* $\Gamma_\lambda : \mathrm{M}(G) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathrm{L}^2(G)))$ *is a completely isometric representation.*

One of the crucial calculations that makes Theorem 3.1 work is that if we let $\{M_\varphi : \varphi \in \mathrm{L}^\infty(G)\}$ be the algebra of multiplication operators on $\mathrm{L}^2(G)$, then for each μ in $\mathrm{M}(G)$ and φ in $\mathrm{L}^\infty(G)$ we have that $\Gamma_\lambda(\mu)M_\varphi = M_{\mu*\varphi}$, so Γ_λ extends the action of $\mathrm{M}(G)$ on $\mathrm{L}^\infty(G)$.

We note that $\Gamma_\lambda(\mathrm{M}(G)) \subset \mathrm{VN}(G) \otimes^{eh} \mathrm{VN}(G)$. In fact, it is shown in [**23**] that $\Gamma_\lambda : \mathrm{M}(G) \to \mathrm{VN}(G) \otimes^{eh} \mathrm{VN}(G)$ is the adjoint of the map $(\Gamma_\lambda)_* : \mathrm{A}(G) \otimes^h \mathrm{A}(G) \to \mathcal{C}_0(G)$ given by $(\Gamma_\lambda)_*(u \otimes v) = u\check{v}$, where $\check{v}(s) = v(s^{-1})$. It follows from Theorem 3.1 that $(\Gamma_\lambda)_*$ is a complete quotient map.

We also note that Neufang has extended Theorem 3.1 to obtain a map from the dual space of left uniformly continuous bounded functions $\mathcal{LUC}(G)^*$ (see [**20**], for information on this algebra) to $\mathcal{CB}(\mathcal{B}(\mathrm{L}^2(G)))$, which extends Γ_λ. See [**22**].

Smith and Spronk [**33**] study the ranges of the maps Γ_π, in general.

Proposition 3.2. (i) $\Gamma_\pi(\mathrm{M}(G)) \subset \mathrm{M}^*_\pi \otimes^{eh} \mathrm{M}^*_\pi \subset \mathrm{MC}^*_\pi \otimes^{eh} \mathrm{MC}^*_\pi$
(ii) $\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^{eh} \mathrm{C}^*_\pi$
(iii) $\Gamma_\pi(\ell^1(G)) \subset \mathrm{D}^*_\pi \otimes^h \mathrm{D}^*_\pi$

We see then that for a discrete measure μ in $\ell^2(G)$ that $\Gamma_\pi(\mu)$ can be approximated uniformly by elementary operators. It is further investigated in [**33**] when this can be done for arbitrary measures.

Theorem 3.3. *The following are equivalent:*
(i) $\Gamma_\pi(\mathrm{M}(G)) \subset \mathrm{M}^*_\pi \otimes^h \mathrm{M}^*_\pi$
(ii) $\Gamma_\pi(\mathrm{L}^1(G)) \subset \mathrm{C}^*_\pi \otimes^h \mathrm{C}^*_\pi$
(iii) $\pi : G \to \mathcal{U}(\mathcal{H})$ *is norm continuous.*

We note that in the case of abelian groups, that (iii) implies (i), above, is shown by Størmer [**37**]. In [**33**] the following fact was used.

Proposition 3.4. *The following are equivalent:*
(i) π *is norm continuous* (ii) C^*_π *is unital* (iii) $\mathrm{C}^*_\pi = \mathrm{D}^*_\pi$.

Condition (iii) allows us to think of norm continuity of π as being a “discreteness” condition. Note that in the case that G is abelian, with dual group $\widehat{G}$, Proposition 3.4 implies that $\mathrm{supp}(\pi)$ is compact. If G is compact, with dual object $\widehat{G}$, then Proposition 3.4 implies that $\mathrm{supp}(\pi)$ is finite. These results are established for the class of direct products of vector groups by compact groups by Kallman [**19**], using different methods.

4. Representations of $\mathrm{M}_{cb}\mathrm{A}(G)$

In [**15**], Herz defines $\mathrm{B}_2(G)$ to be the space of Schur multipliers u on $\mathcal{B}(\ell^2(G))$ which are continuous and satisfy the following invariance condition: $u(s,t) = u(sr,tr)$ for all s,t,r in G. Bożejko and Fendler [**5**] prove that $\mathrm{B}_2(G) = \mathrm{M}_{cb}\mathrm{A}(G)$. There are similar results, due to Haagerup [**14**] and Jolissaint [**18**]. Thus there is a strong link between $\mathrm{M}_{cb}\mathrm{A}(G)$ and Schur multipliers.

The main result relating $\mathrm{M}_{cb}\mathrm{A}(G)$ with measurable Schur multipliers follows. It can be found in the present form in [**34, 35**], but there is an isometric version due to Haagerup [**13**]. Below, we let m denote the left Haar measure on G and

$$\mathrm{V}^\infty_{\mathrm{inv}}(G) = \left\{ w \in \mathrm{V}^\infty(G,m) : \begin{array}{c} w(s,t) = w(sr,tr) \text{ for l.m.a.e.} \\ (s,t) \text{ in } G\times G \text{ and all } r \text{ in } G \end{array} \right\}$$

where l.m.a.e. means "locally marginally almost every".

Theorem 4.1. *If $u \in \mathrm{M}_{cb}\mathrm{A}(G)$, then Nu, given for (s,t) in $G\times G$ by*

$$Nu(s,t) = u(st^{-1})$$

defines an element of $\mathrm{V}^\infty(G,m)$. Moreover the map $N : \mathrm{M}_{cb}\mathrm{A}(G) \to \mathrm{V}^\infty(G,m)$ is a complete isometry whose image is $\mathrm{V}^\infty_{\mathrm{inv}}(G)$.

Using slice maps on each Nu, it can be established that

$$N(\mathrm{M}_{cb}\mathrm{A}(G)) \subset \mathrm{V}^b(G) = \mathcal{C}_b(G) \otimes^{eh} \mathcal{C}_b(G)$$

which is an analogue of Proposition 3.2 (i). However, we can further establish that

$$N(\mathrm{M}_{cb}\mathrm{A}(G)) \subset \overline{\mathrm{M}_{cb}\mathrm{A}(G)}^{\|\cdot\|_\infty} \otimes^{eh} \overline{\mathrm{M}_{cb}\mathrm{A}(G)}^{\|\cdot\|_\infty} \subset \mathcal{WAP}(G) \otimes^{eh} \mathcal{WAP}(G)$$

where $\mathcal{WAP}(G)$ is the algebra of weakly almost periodic functions on G. We note that the inclusion $\mathrm{M}_{cb}\mathrm{A}(G) \subset \mathcal{WAP}(G)$ was established by Xu [**42, 43**].

We note that as a Corollary of Theorem 4.1, we obtain a characterisation of the predual $\mathrm{Q}(G)$ of $\mathrm{M}_{cb}\mathrm{A}(G)$, whose existence was established by de Cannière and Haagerup [**7**]: there is a map from $\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G)$ to $\mathrm{Q}(G)$ given on elementary tensors by $f \otimes g \mapsto f * g$, which is a complete quotient. Thus it follows from Losert's result characterising when $\mathrm{M}_{cb}\mathrm{A}(G) = \mathrm{B}(G)$ that $\mathrm{C}^*(G)$ is a quotient of $\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G)$ exactly when G is amenable. An equivalent form of this result appears to go back to Grothendieck, and is exposed by Racher [**28**].

5. Parallels

We now strive to compare the results of the previous two sections. Below we identify $\mathrm{L}^\infty(G)$ with the algebra of multiplication operators on $\mathrm{L}^2(G)$. If $\mathcal{M}$ is an von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, we denote the space of completely bounded normal $\mathcal{M}$-bimodule maps on $\mathcal{B}(\mathcal{H})$ by $\mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(\mathcal{H}))$. Note that by Theorem 2.3 (iii) this is always $\mathcal{M}' \otimes^{eh} \mathcal{M}'$. Moreover, also by Theorem 2.3 (iii), if $\mathcal{M}$ is locally cyclic, then $\mathcal{CB}^\sigma_{\mathcal{M}}(\mathcal{B}(\mathcal{H})) = \mathcal{B}^\sigma_{\mathcal{M}}(\mathcal{B}(\mathcal{H}))$.

Let $\rho : G \to \mathcal{U}(\mathrm{L}^2(G))$ be the right regular representation. Since ρ is similar to the left regular representation λ, we have that Theorem 3.1 holds for $\Gamma_\rho : \mathrm{M}(G) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathrm{L}^2(G)))$.

Theorem 5.1. *The ranges of Γ_ρ and N can be characterised by the following invariance conditions:*

(i) [**22**] $\Gamma_\rho(\mathrm{M}(G)) = \left\{T \in \mathcal{CB}^\sigma_{\mathrm{VN}(G)}(\mathcal{B}(\mathrm{L}^2(G))) : T\mathrm{L}^\infty(G) \subset \mathrm{L}^\infty(G)\right\}$.

(ii) [**23**] $N(\mathrm{M}_{cb}\mathrm{A}(G)) = \left\{T \in \mathcal{CB}^\sigma_{\mathrm{L}^\infty(G)}(\mathcal{B}(\mathrm{L}^2(G))) : T\mathrm{VN}(G) \subset \mathrm{VN}(G)\right\}$.

This is very strong evidence that Theorems 3.1 and 4.1 are two examples of the same phenomenon – to wit, there may be a Kac algebra or quantum group theorem from which these results can be deduced. This becomes even more apparent if we consider the case that G is abelian, with dual group $\widehat{G}$. In analogy with maps ρ, N and Γ_ρ above, we define $\hat{\rho} : \widehat{G} \to \mathcal{U}(\mathrm{L}^2(\widehat{G}))$, $\widehat{N} : \mathrm{B}(\widehat{G}) = \mathrm{M}_{cb}\mathrm{A}(\widehat{G}) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathrm{L}^2(\widehat{G})))$ and $\widehat{\Gamma}_{\hat{\rho}} : \mathrm{M}(\widehat{G}) \to \mathcal{CB}^\sigma(\mathcal{B}(\mathrm{L}^2(\widehat{G})))$. We can adjust the Plancherel unitary $U : \mathrm{L}^2(G) \to \mathrm{L}^2(\widehat{G})$ in such a way that

$$\mathrm{Ad}U(\mathrm{VN}(G)) = \mathrm{L}^\infty(\widehat{G}) \quad \text{and} \quad \mathrm{Ad}U(\mathrm{L}^\infty(G)) = \mathrm{VN}(\widehat{G})$$

where $\mathrm{Ad}Ux = UxU^*$ for each x in $\mathcal{B}(\mathrm{L}^2(G))$. The following is in [**23**].

Theorem 5.2. (i) $\mathrm{Ad}U\Gamma_\rho(\mathrm{M}(G))\mathrm{Ad}U^* = \widehat{N}(\mathrm{B}(\widehat{G}))$

(ii) $\mathrm{Ad}UN(\mathrm{B}(G))\mathrm{Ad}U^* = \widehat{\Gamma}_{\hat{\rho}}(\mathrm{M}(\widehat{G}))$

Let us close by illustrating a final parallel. If G is discrete, then it follows from Proposition 3.2 (iii) that

$$\Gamma_\lambda(\ell^1(G)) \subset \mathrm{C}^*_r(G) \otimes^h \mathrm{C}^*_r(G)$$

where $\mathrm{C}^*_r(G) = \mathrm{C}^*_\lambda$ is the reduced C*-algebra of G. An analogue ought to hold then for compact groups in the context of the Fourier algebra. It is shown in [**34, 35**] that the canonical map from G to its almost periodic compactification G^{ap} induces a complete isometry $\mathrm{A}(G^{\mathrm{ap}}) \hookrightarrow \mathrm{M}_{cb}\mathrm{A}(G)$. The result (i), below, is due to Spronk and Turowska [**36**] and its extension, (ii), is in [**34**]. If G is a compact abelian group, then (i) is due to Varopoulos [**38**]. An equivalent result to (ii) has also been discovered by Racher [**29**].

Theorem 5.3. (i) *If G is compact then* $N(\mathrm{A}(G)) \subset \mathrm{V}_0(G) = \mathcal{C}_b(G) \otimes^h \mathcal{C}_b(G)$.
(ii) $N(\mathrm{A}(G^{\mathrm{ap}})) \subset \mathcal{C}_b(G) \otimes^h \mathcal{C}_b(G)$

We note that in analogy with Theorem 3.3, it has been shown in [**36, 34, 35**] that the inclusion $N(\mathrm{A}(G)) \subset \mathrm{V}_0(G) = \mathcal{C}_0(G) \otimes^h \mathcal{C}_0(G)$ implies that G is compact.

References

[1] D. P. Blecher. Geometry of the tensor product of C*-algebras. *Math. Proc. Cambridge Phil. Soc.*, 104(1):119–127, 1988.

[2] D. P. Blecher. The standard dual of an operator space. *Pacific Math. J.*, 153(1):15–30, 1992.

[3] D. P. Blecher and R. R. Smith. The dual of the Haagerup tensor product. *J. London Math. Soc.*, 45(2):126–144, 1992.

[4] M. Bożejko. Positive definite bounded matrices and a characterization of amenable groups. *Proc. Amer. Math. Soc.*, 95(3):357–360, 1985.

[5] M. Bożejko and G. Fendler. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A (6)*, 3(2):297–302, 1984.

[6] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank 1. *Invent. Math.*, 96:507–549, 1989.

[7] J. de Cannière and U. Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.*, 107:455–500, 1984.

[8] E. G. Effros and Z.-J. Ruan. A new approach to operator spaces. *Canad. Math. Bull.*, 34:329–337, 1991.

[9] E. G. Effros and Z.-J. Ruan. *Operator Spaces*, volume 23 of *London Math. Soc., New Series.* Claredon Press, Oxford Univ. Press, New York, 2000.

[10] E. G. Effros and Z.-J. Ruan. Operator convolution algebras: an approach to quantum groups. To appear in *J. Operator Theory.*, 2002.

[11] P. Eymard. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[12] F. Ghahramani. Isometric representations of M(G) on $\mathcal{B}(\mathcal{H})$. *Glasgow Math. J.*, 23:119–122, 1982.

[13] U. Haagerup. Decomposition of completely bounded maps on operator algebras. Unpublished, 1980.

[14] U. Haagerup. Group C*-algebras without the completely bounded approximation property. Unpublished, 1988.

[15] C. S. Herz. Une généralisation de la notion transformée de Fourier-Stieltjes. *Ann. Inst. Fourier (Grenoble)*, 24(3):145–157, 1974.

[16] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*, volume 115 of *Die Grundlehern der mathemarischen Wissenschaften*. Springer, New York, second edition, 1979.

[17] M. Hladnik. Compact Schur multipliers. *Proc. Amer. Math. Soc.*, 128:2585–2591, 2000.

[18] P. Jolissaint. A characterization of completely bounded multipliers of the Fourier algebras. *Colloq. Math.*, 63:311–313, 1992.

[19] R. R. Kallman. A characterization of uniformly continuous unitary representations of connected locally compact groups. *Michigan Math. J.*, 16:257–263, 1969.

[20] A. T.-M. Lau. Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups. *Math. Proc. Camb. Phil. Soc.*, 99:273–283, 1986.

[21] V. Losert. Properties of the Fourier algebra that are equivalent to amenability. *Proc. Amer. Math. Soc.*, 92(3):347–353, 1984.

[22] M. Neufang. Isometric representations of convolution algebras as completely bounded module homomorphisms and a characterization of the measure algebra. Unpublished, 2001.

[23] M. Neufang, Z.-J. Ruan, and N. Spronk. Completely isometric representations of measure algebras and of completely bounded multipliers of Fourier algebras. In preparation, 2004.

[24] T. W. Palmer. *Banach Algebras and the General Theory of* $*$*-algebras, Vol. I Algebras and Banach Algebras*, volume 49 of *Encyclopedia of Mathematics and it Applications*. Cambridge University Press, Cambridge, New York, 1994.

[25] A. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[26] V. I. Paulsen. Every completely polynomially bounded operator is similar to a contraction. *J. Funct. Anal.*, 55:1–17, 1984.

[27] G. Pisier. Multipliers and lacunary sets in non-amenable groups. *Amer. J. Math.*, 117:337–376, 1995.

[28] G. Racher. A proposition of Grothendieck revisited. In *Banach Space Theory and Its Applications*, volume 991 of *Lec. Notes in Math.*, pages 215–227, Berlin Heidelberg, 1983. Springer.

[29] G. Racher. The nuclear multipiers from $L^1(G)$ into $L^\infty(G)$. *J. Funct. Anal.*, 122:279–306, 1994.

[30] V. Runde. *Lectures on Amenability*, volume 1774 of *Lec. Notes in Math.* Springer, Berlin Heidelberg, 2002.

[31] V. Shulman and L. Turowska. Operator synthesis. 1. synthetic sets, bilattices and tensor algebras. Preprint. To appear in *J. Funct. Anal.* Also see ArXiv `math.FA/030820`, 2003.

[32] R. R. Smith. Completely bounded module maps and the Haagerup tensor product. *J. Funct. Anal.*, 102(1):156–175, 1991.

[33] R. R. Smith and N. Spronk. Representations of group algebras in spaces of completely bounded maps. Preprint. To appear in *Indiana Univ. Math. J.* Also see ArXiv `math.FA/0310421`, 2003.

[34] N. Spronk. *On Multipliers of the Fourier Algebra of a Locally Compact Group.* PhD thesis, University of Waterloo, Waterloo, Ontario, Canada, 2002.

[35] N. Spronk. Measurable Schur multipliers and completely bounded multipliers of the Fourier algebra. To appear in *Proc. London Math. Soc.* Also see ArXiv `math.FA/0210304`, 2003.

[36] N. Spronk and L. Turowska. Spectral synthesis and operator synthesis for compact groups. *J. London Math. Soc.*, 66:361–376, 2002.

[37] E. Størmer. Regular Abelian Banach algebras of linear maps of operator algebras. *J. Funct. Anal.*, 37:331–373, 1980.

[38] N. Th. Varopoulos. Tensor algebras and harmonic analysis. *Acta. Math.*, 119:51–112, 1967.

[39] M. E. Walter. Dual algebras. *Math. Scand.*, 58:77–104, 1986.

[40] M. E. Walter. On a new method for defining the norm of Fourier-Stietjes algebras. *Pacific J. Math.*, 137:209–223, 1989.

[41] J. G. Wendell. Left centralizers and isomorphisms of group algebras. *Pacific J. Math.*, 2:251–261, 1952.

[42] G. Xu. *Multipiers of the Algebras $A_p(G)$.* PhD thesis, SUNY Buffalo, NY, 1995.

[43] G. Xu. Herz-Schur multipliers and weakly almost periodic functions on locally compact groups. *Trans. Amer. Math. Soc.*, 349:2525–2536, 1997.