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Journal of Functional Analysis 203 (2003) 286–304

JOURNAL OF Functional Analysis

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Ideals with bounded approximate identities in Fourier algebras

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Received 10 July 2002; accepted 23 September 2002

Abstract

We make use of the operator space structure of the Fourier algebra $A(G)$ of an amenable locally compact group to prove that if H is any closed subgroup of G, then the ideal $I(H)$ consisting of all functions in $A(G)$ vanishing on H has a bounded approximate identity. This result allows us to completely characterize the ideals of $A(G)$ with bounded approximate identities. We also show that for several classes of locally compact groups, including all nilpotent groups, $I(H)$ has an approximate identity with norm bounded by 2, the best possible norm bound.

Let G be a locally compact group and $A(G)$ the Fourier algebra of G as introduced by Eymard [6]. $A(G)$ is the linear subspace of $C_0(G)$ consisting of all coordinate functions of the regular representation of G and can be identified with the predual of the group von Neumann algebra $VN(G)$ generated by left translations on $L^2(G)$. With the norm given by this duality and pointwise multiplication, $A(G)$ becomes a commutative Banach algebra. The spectrum of $A(G)$ is G (point evaluation of functions in $A(G)$). Leptin [20] has shown that $A(G)$ has a bounded approximate identity precisely when G is amenable.

One of the most interesting open problems in the ideal theory of $A(G)$ is to determine the closed ideals of $A(G)$ with bounded approximate identities. Recall that when G is abelian, $A(G)$ is isometrically isomorphic (by means of the inverse Fourier

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[1] Supported by Grants from NSERC Canada.

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doi:10.1016/S0022-1236(02)00121-0

transform) to $L^1(\hat{G})$, the L^1-convolution algebra on the dual group $\hat{G}$ of G. Now, for an abelian locally compact group Γ the closed ideals of $L^1(\Gamma)$ with bounded approximate identities have been completely characterized by Liu et al. [21]. This characterization suggests to conjecture that a closed ideal I of $A(G)$ has a bounded approximate identity if and only if I equals

$$I(E) = \{u \in A(G) : u(x) = 0 \text{ for all } x \in E\}$$

for some set E in the (closed) coset ring of G. This conjecture has been verified in [8] for amenable groups with small conjugation invariant neighbourhoods of the identity (so-called SIN-groups).

In this paper we shall first use the fact that the Fourier algebra has an additional structure as an operator space to show that if G is amenable and H is any closed subgroup of G, then $I(H)$ has a bounded approximate identity (Theorem 1.5). This allows us to establish the above conjecture for arbitrary amenable locally compact groups (Theorem 2.3). Some of these results extend to the more general Herz–Figà–Talamanca algebras $A_p(G), 1 < p < \infty$ (Theorem 4.3).

Using results from Delaporte and Derighetti [2], it can be shown that 2 is the best possible norm bound for an approximate identity of $I(H)$ as long as H is not open in G (see [16]). In [16] it was shown that the norm bound 2 is achieved if the pair (G, H) satisfies a certain separation property of positive definite functions. This property is satisfied in SIN-groups G for all closed subgroups H, but most likely fails in any other class of locally compact groups (see [16,18]). We proceed to show that, nevertheless, if G is a locally nilpotent group or an almost connected group with a compact normal subgroup K such that G/K is nilpotent, then indeed $I(H)$ has an approximate identity with norm bounded by 2 for every closed subgroup H of G (Theorems 3.3 and 3.7). The question of whether this holds true for all amenable groups, remains open.

0. Some preliminaries

Let A be a commutative Banach algebra. A net $\{u_\alpha\}$ in A is called a *bounded approximate identity* if $||u_\alpha u - u|| \to 0$ for every $u \in A$ and if there exists $c > 0$ such that $||u_\alpha|| \leqslant c$ for all α. We refer to c as a *norm bound* of the net $\{u_\alpha\}$.

An operator space is a vector space V together with a family $\{|| \cdot ||_n\}$ of norms on $M_n(V)$, the vector space of $n \times n$ matrices with entries in V such that

(i) $\left|\left|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right|\right|_{n+m} = \max\{||A||_n, ||B||_m\}$ for each $A \in M_n(V)$ and $B \in M_m(V)$, and
(ii) $||[a_{ij}]A[b_{ij}]||_n \leqslant ||[a_{ij}]|| \cdot ||A||_n \cdot ||[b_{ij}]||$ for all $[a_{ij}], [b_{ij}] \in M_n(\mathbb{C})$ and $A \in M_n(V)$.

Let X and Y be operator spaces and let $T : X \to Y$ be linear. For each $n \in \mathbb{N}$, we define $T^{(n)} : M_n(X) \to M_n(Y)$ by $T^{(n)}([a_{ij}]) = [T(a_{ij})]$. T is said to be *completely* *bounded* if

$$\|T\|_{\rm cb} = \sup_{n \in \mathbb{N}} \{\|T^{(n)}\|\} < \infty.$$

Moreover, T is said to be *completely contractive* if $\|T\|_{\rm cb} \leqslant 1$.

If V is a linear subspace of $\mathscr{B}(\mathscr{H})$, the C^*-algebra of bounded operators on a Hilbert space, then V is an operator space with the matricial norms inherited from $M_n(V) \subseteq M_n(\mathscr{B}(\mathscr{H}))$ identified with $\mathscr{B}(\mathscr{H}^n)$. In fact, every operator space V is completely isometric to a vector subspace of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ [5, Theorem 2.3.5]. If X and Y are subspaces of $\mathscr{B}(\mathscr{H})$, then a linear map $T : X \to Y$ is called *completely positive* if each $T^{(n)}$ is a positive map. Concerning all these notions we refer the reader to [5,23].

Let G be a locally compact group and let $P(G)$ denote the set of all continuous positive definite functions on G. The linear span of $P(G)$, $B(G)$, forms an algebra called the Fourier–Stieltjes algebra of G. Then $B(G)$ can be identified with the dual space of the group C^*-algebra $C^*(G)$ (see [6]). Let $A(G)$ denote the closed ideal of $B(G)$ generated by all functions with compact support in $B(G)$. As shown in [6], $A(G)$ consists of all functions of the form $u(x) = \sum_{n=1}^{\infty} (f_n * \tilde{g}_n)(x^{-1})$, where $f_n, g_n \in L^2(G)$ and $\sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 < \infty$. Here $\tilde{f}$ is defined by $\tilde{f}(x) = \overline{f(x^{-1})}$. The norm on $A(G)$ is given by $\|u\| = \inf\{\sum_{n=1}^{\infty} \|f_n\|_2\|g_n\|_2 : u(x) = \sum_{n=1}^{\infty} (f_n * \tilde{g}_n(x^{-1})\}$ and $A(G)$ can be identified with the predual of the von Neumann algebra $VN(G)$ generated by the operators $\rho(x)$, $x \in G$, on $L^2(G)$.

Since $A(G)$ is an ideal in $B(G)$ and $VN(G) = A(G)^*$, there is a natural action of $B(G)$ on $VN(G)$ given by

$$\langle u \cdot T, v \rangle = \langle T, uv \rangle$$

for $v \in A(G)$ and $u \in B(G)$. We will say that a linear map $P : VN(G) \to VN(G)$ commutes with the action of $A(G)$ if $P(u \cdot T) = u \cdot P(T)$ for all $T \in VN(G)$ and $u \in A(G)$.

Since $VN(G)$ is a von Neumann algebra, its predual $A(G)$ inherits a natural structure as an operator space with respect to which it becomes a completely contractive Banach algebra. That is, the multiplication on $A(G)$ extends to a completely contractive map $m : A(G) \hat{\otimes} A(G) \to A(G)$ with $m(u \otimes v) = uv$. Here $A(G) \hat{\otimes} A(G)$ denotes the operator space analogue of the projective tensor product. This allows us to apply operator space techniques to study problems concerning ideals in $A(G)$. In particular, we shall make direct use of Ruan's remarkable generalization of Johnson's characterization of amenable group algebras, namely that G is amenable as a locally compact group if and only if $A(G)$ is operator amenable. We refer the reader to [5,24] for details.

Let I be a closed ideal in $A(G)$. Then $I^{\perp} = \{T \in VN(G) : \langle T, u \rangle = 0$ for each $u \in I\}$ is an $A(G)$-submodule of $VN(G)$. Let H be a closed subgroup of G, and define $VN_H(G)$ to be the von Neumann subalgebra of $VN(G)$ generated by $\{\rho(x) : x \in H\}$.

Then $I(H)^{\perp} = VN_H(G)$ (see, for example [16, p. 101]). Moreover, $VN_H(G)$ is known to be $*$-isomorphic to $VN(H)$.

1. The ideals $I(H)$

In this section we shall use the operator space structure of the Fourier algebra $A(G)$ of an amenable group G to show that for each closed subgroup H of G the ideal $I(H)$ has a bounded approximate identity.

Lemma 1.1. *Let H be an amenable locally compact group. Then there exists a completely contractive projection $P : \mathscr{B}(L^2(H)) \to VN(H)$.*

Proof. First note that $VN(H)$ is the commutant of $\{\rho(x) : x \in H\}$, where $\rho : H \to \mathscr{B}(L^2(H))$ is the right regular representation of H. Let m be a left invariant mean on $L^\infty(H)$. It will be convenient for us to regard m as a finitely additive measure on H. Now define $P : \mathscr{B}(L^2(H)) \to \mathscr{B}(L^2(H))$ to be the weak operator converging integral

$$P(T) = \int_H \rho(s) T \rho(s)^* \, dm(s)$$

for each $T \in \mathscr{B}(L^2(H))$. That is,

$$\langle P(T)f, g \rangle = m(s \to \langle \rho(s) T \rho(s)^* f, g \rangle)$$

for each $f, g \in L^2(H)$.

From the invariance of m, it is easy to see that for each $T \in \mathscr{B}(L^2(H))$,

$$\rho(t) P(T) = P(T) \rho(t)$$

for all $t \in H$. It follows that $P(T) \in VN(H)$. It is also clear that if $T \in VN(H)$, then $P(T) = T$.

We have that P is completely positive, since each amplification

$$P^{(n)} : M_n(\mathscr{B}(L^2(H))) \to M_n(VN(H))$$

is given by the weak operator converging integral

$$P^{(n)}([T_{ij}]) = \int_H \mathrm{diag}(\rho(s))[T_{ij}]\mathrm{diag}(\rho(s))^* \, dm(s),$$

for each $[T_{ij}] \in M_n(\mathscr{B}(L^2(H)))$, where $\mathrm{diag}(\rho(s))$ denotes the $n \times n$ diagonal matrix with all diagonal entries equal to $\rho(s)$. Finally, since $P(I) = I$, we get that P is completely contractive. $\square$

Assume that H is an amenable group. We claim that if $\mathcal{M}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ such that $\mathcal{M}$ is $*$-isomorphic to $VN(H)$, then there is a contractive completely positive expectation $\tilde{P}: \mathcal{B}(\mathcal{H}) \to \mathcal{M}$. To see this let $\Phi: VN(H) \to \mathcal{M}$ be a $*$-isomorphism. The Arveson-Wittstock Extension Theorem (see [5, Theorem 4.1.5]) implies that $\Phi^{-1}: \mathcal{M} \to VN(H)$ admits a completely contractive extension $\Psi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(L^2(H))$. Moreover, since $\Psi(I) = I$, Ψ is completely positive [5, Corollary 5.1.2]. We now let $\tilde{P} = \Phi \circ P \circ \Psi$ where P is the projection constructed in Lemma 1.1.

Proposition 1.2. *Let G be a locally compact group and let H be an amenable closed subgroup of G. Then there exists a completely contractive projection from $VN(G)$ onto $I(H)^{\perp}$.*

Proof. We recall that $I(H)^{\perp} = VN_H(G)$ and that $VN_H(G)$ is $*$-isomorphic to $VN(H)$. It follows from Lemma 1.1 and the discussion preceding this proposition that there exists a completely contractive projection

$$\tilde{P}: \mathcal{B}(L^2(G)) \to VN_H(G) = I(H)^{\perp}.$$

We now simply restrict $\tilde{P}$ to $VN(G)$. □

For G σ-compact and H an amenable closed subgroup, the existence of a continuous projection from $VN(G)$ onto $VN(H)$ was established by Lohoué [22]. For G not necessarily σ-compact and H a closed amenable subgroup, the existence of a contractive projection from $VN(G)$ onto $VN(H)$ was obtained by Derighetti [4]. However, for our purposes, we will need the additional fact provided by Proposition 1.2 that the projection can be chosen to be a complete contraction.

Theorem 1.3. *Let G be an amenable locally compact group and H a closed subgroup of G. Then there exists a completely bounded projection $P: VN(G) \to I(H)^{\perp}$ such that*

$$P(u \cdot T) = u \cdot P(T)$$

for all $T \in VN(G)$ and $u \in A(G)$.

Proof. Since G is amenable, the Fourier algebra $A(G)$ is an operator amenable completely contractive Banach algebra. By Proposition 1.2, there is a completely contractive projection from $VN(G)$ onto $I(H)^{\perp}$. The existence of an invariant completely bounded projection now follows immediately from [28, Theorem 3]. □

Let G be a locally compact group and let $\mathcal{M}$ be an invariant W^*-subalgebra of $VN(G)$. Let $H = \{x \in G : \rho(x) \in \mathcal{M}\}$. Then H is a closed subgroup of G and $\mathcal{M} = I(H)^{\perp}$ [26, Theorem 3]. In [19] it was shown that if H is normal or, more generally, G has the H-separation property, then $\mathcal{M}$ is invariantly complemented, that is, there exists a projection from $VN(G)$ onto $\mathscr{M}$ commuting with the action of $A(G)$. In particular, if G is an SIN-group, then every invariant W^*-subalgebra of $VN(G)$ is invariantly complemented. Theorem 1.3 now provides another class of locally compact groups sharing this property:

Corollary 1.4. *Let G be an amenable locally compact group. Then any invariant W^*-subalgebra of $VN(G)$ is invariantly complemented.*

Theorem 1.5. *Let G be an amenable locally compact group and let H be a closed subgroup of G. Then the ideal $I(H)$ has a bounded approximate identity.*

Proof. Theorem 1.3 establishes the existence of an invariant projection P of norm 1 from $VN(G)$ onto $I(H)^{\perp}$. Since G is amenable, the existence of such a projection allows to construct a bounded approximate identity in $I(H)$. This follows from either [7, Proposition 6.4 and its proof] or [12, Proposition 11]. □

We note that in Theorem 1.5 we have focused only on amenable locally compact groups. It is worth noting that this is really not a restriction as the following improvement of [8, Corollary 3.15] shows.

Corollary 1.6. *Let G be a locally compact group. Then the following are equivalent:*

(i) *G is amenable.*
(ii) *$I(H)$ has a bounded approximate identity for some proper closed subgroup H of G.*
(iii) *$I(H)$ has a bounded approximate identity for every proper closed subgroup H of G.*

2. Complete characterization

In this section we are going to show that Theorem 1.5 leads to a complete characterization of all closed ideals with bounded approximate identities in $A(G)$ when G is an amenable locally compact group.

The following standard facts will be used throughout. Let I and J be closed ideals of a commutative Banach algebra A. Then, if I and J both have bounded approximate identities, then so do $I \cap J$ and $\overline{I+J}$. Also, if I and A/I both have bounded approximate identities, then A has a bounded approximate identity.

Lemma 2.1. *Let G be an amenable locally compact group and let H and K be closed subgroups of G such that $K \subseteq H$ and K is open in H. Then $I(aH \backslash bK)$ has a bounded approximate identity for any $a, b \in G$.*

Proof. Since $aH \backslash bK = a(H \backslash a^{-1}bK)$ and $H \backslash a^{-1}bK \neq H$ only if $a^{-1}b \in H$ and in this case $H \backslash a^{-1}bK = a^{-1}b(H \backslash K)$, it suffices to show that $I(H \backslash K)$ has a bounded approximate identity. Let

$$J = \{u \in A(H) : u(x) = 0 \text{ for all } x \in H \backslash K\}.$$

Since K is open in H, the map $u \to u|_K$ is an isometric isomorphism from J onto $A(K)$. Since K is amenable, $A(K)$ has a bounded approximate identity and hence so does J. With $r : A(G) \to A(H)$ denoting the restriction map, $I(H\backslash K) = r^{-1}(J)$. Thus, since $I(H)$ has a bounded approximate identity (Theorem 1.5), $I(H\backslash K)$ has one provided that $r^{-1}(J)/I(H)$ does so. Now, by Forrest [7, Lemma 3.8], the mapping $\phi : u + I(H) \to u|_H$ is an isometric isomorphism between $A(G)/I(H)$ and $A(H)$. It follows that, since J has a bounded approximate identity, so does $\phi^{-1}(J) = r^{-1}(J)/I(H)$. $\square$

For any (discrete) group D, let $\mathscr{R}(D)$ denote the Boolean ring of subsets of D generated by all left cosets of subgroups of D. $\mathscr{R}(D)$ is called the *coset ring* of D. Now, let G be a locally compact group and let G_d denote the group G with the discrete topology. Define the *closed coset ring* of G by

$$\mathscr{R}_c(G) = \{E \in \mathscr{R}(G_d) : E \text{ is closed in } G\}.$$

When G is abelian, the sets in $\mathscr{R}_c(G)$ have been described, independently, by Gilbert [11] and Schreiber [25]. Forrest [8] has verified the analogous description for arbitrary G. The result can be reformulated as follows. $\mathscr{R}_c(G)$ is the collection of all subsets E of G of the form

$$E = \bigcup_{i=1}^{n} \left(a_i H_i \Bigg\backslash \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where $a_i, b_{i,j} \in G$, H_i is a closed subgroup of G and $K_{i,j}$ is an open subgroup of H_i ($n, m_i \in \mathbb{N}_0, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m_i$).

For a closed ideal I of $A(G)$, let

$$Z(I) = \{x \in G : u(x) = 0 \text{ for all } u \in I\}.$$

Recall that a closed subset E of G is a *set of synthesis* or *spectral set* if $I(E)$ is the only closed ideal J of $A(G)$ with $Z(J) = E$. It is easy to see that E is a set of synthesis if and only if every $T \in VN(G)$ with supp $T \subseteq E$ belongs to $I(E)^{\perp}$ (cf. [17, Lemma 2.5]). By Takesaki and Tatsuuma [26, Theorem 3] each closed subgroup H of G is a set of synthesis, and hence so is every coset aH, $a \in G$.

Lemma 2.2. *Let G be an amenable locally compact group. Then, for each $E \in \mathscr{R}_c(G)$,*

(i) *E is a set of synthesis for $A(G)$,*

(ii) *$I(E)$ has a bounded approximate identity.*

Proof. Assume first that E is of the form $E = aH \setminus \bigcup_{j=1}^{m} c_j K_j$, where H is a closed subgroup of G, $m \in \mathbb{N}_0$, $a, c_j \in G$ and the K_j are open subgroups of H, $1 \leqslant j \leqslant m$. Then E is a set of synthesis since aH is of synthesis and E is open and closed in aH (see [27, Theorem 4]). To see that $I(E)$ has a bounded approximate identity, consider the ideal

$$J = \overline{\sum_{j=1}^{m} I(aH \setminus c_j K_j)}$$

of $A(G)$. It follows from Lemma 2.1 and preceding remarks that J has a bounded approximate identity. Now,

$$Z(J) = \bigcap_{j=1}^{m} (aH \setminus c_j K_j) = E,$$

and hence $J = I(E)$ since E is a spectral set. This finishes the proof for such E.

Next, let $E = \bigcup_{i=1}^{n} E_i$ where each E_i is of the above form. We prove by induction on n that E is a set of synthesis and $I(E)$ has a bounded approximate identity. The case $n = 1$ is proved in the first paragraph. Let $n \geqslant 2$ and $F = \bigcup_{i=1}^{n-1} E_i$, and suppose that F is a set of synthesis and that $I(F)$ has a bounded approximate identity. Clearly, then $I(E) = I(F) \cap I(E_n)$ has a bounded approximate identity. Finally, recall that, for any Tauberian regular commutative Banach algebra A with structure space $\Delta(A)$, the union of two Ditkin sets in $\Delta(A)$ is a Ditkin set [27, Theorem 1]. Clearly, E_n and F are Ditkin sets since they both are spectral sets and the ideals $I(E_n)$ and $I(F)$ possess approximate identities. Thus we in particular obtain that E is a spectral set. □

Theorem 2.3. *Let G be a amenable locally compact group. Let I be a closed ideal of $A(G)$. Then I has a bounded approximate identity if and only if $I = I(E)$ for a subset E of G of the form*

$$E = \bigcup_{i=1}^{n} \left(a_i H_i \Bigg\backslash \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where $a_i, b_{i,j} \in G$, H_i is a closed subgroup of G and $K_{i,j}$ is an open subgroup of H_i $(n, m_i \in \mathbb{N}_0, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m_i)$.

Proof. If E is of the above form, then $I(E)$ has a bounded approximate identity by Lemma 2.2. Conversely, suppose that I is a closed ideal of $A(G)$ with bounded approximate identity and let $E = Z(I)$. Then $E \in \mathscr{R}_c(G)$. This was proved in [7] in case $I = I(E)$. However, the proof of Lemma 3.3 of [7] actually shows that if J is any closed ideal of $A(G)$ with bounded approximate identity, then the characteristic function of $Z(J)$ belongs to the Fourier–Stieltjes algebra of G_d. The rest amounts to an application of Host's generalization [15] of Cohen's idempotent theorem (see

[7, Proposition 3.5]). Since, by Lemma 2.2, E is a set of synthesis, it follows that $I = I(E)$, as required. □

In [10, Theorem 8.5] it was shown that if A is an operator amenable completely contractive Banach algebra and I is a closed ideal in A, then I is operator amenable if and only if I has a bounded approximate identity. The following corollary, which improves [10, Theorem 8.6], is now an immediate consequence of the preceding theorem.

Corollary 2.4. *Let G be an amenable locally compact group and let I be a closed ideal of $A(G)$. Then I is operator amenable if and only if $I = I(E)$ for some $E \in \mathscr{R}_c(G)$.*

3. Best possible norm bound

Let H be a closed subgroup of a locally compact group G. If G/H is infinite, then 2 is the smallest possible norm bound for an approximate identity in the ideal $I(H)$ (see [2,16, Theorem 3.4]). An approximate identity with norm bound 2 exists whenever G is amenable and there exists a projection of norm 1 from $VN(G)$ onto $VN_H(G)$ commuting with the action of $A(G)$. As shown in [16], such a projection in turn exists if the pair (G, H) has the following separation property of positive definite functions, which at least holds in SIN-groups (see [16]).

Let G be a locally compact group and H a closed subgroup of G. Let

$$P_H(G) = \{\varphi \in P(G) : \varphi(y) = 1 \text{ for all } y \in H\}.$$

We say that G has the *H-separation property* if given any $x \in G \backslash H$, there exists $\varphi \in P_H(G)$ such that $\varphi(x) \neq 1$. This property has been extensively studied in [16,18].

Now, norm 1 projections as above also exist for other classes of locally compact groups. To show this is our goal in the present section. The next two lemmas are basic tools for that. Note, however, that if H and K are closed subgroups of G such that $H \subseteq K$ and if K has the H-separation property and G has the K-separation property, then G need not have the H-separation property.

Lemma 3.1. *Let G be a locally compact group and let H and K be closed subgroups of G such that $H \subseteq K$. Suppose there exist completely positive (respectively, completely contractive) norm one projections $P : VN(G) \to VN_K(G)$ and $Q : VN(K) \to VN_H(K)$ commuting with the action of $A(G)$ and $A(K)$, respectively. Then there exists a completely positive (respectively, completely contractive) projection of norm one from $VN(G)$ onto $VN_H(G)$ commuting with the action of $A(G)$.*

Proof. For closed subgroups C and D of G such that $D \subseteq C$ let $r_{C,D} : A(C) \to A(D)$ be the restriction map and $r^*_{C,D} : VN(D) \to VN(C)$ the dual map. Thus $\langle r^*_{C,D}(S), u \rangle = \langle S, u|_D \rangle$ for all $S \in VN(D)$ and $u \in A(C)$ and $r^*_{C,D}$ is an isomorphism between $VN(D)$

and $VN_D(C)$ (see [13, Lemma 3.1]). Let $j_{C,D} : VN_D(C) \to VN(D)$ denote its inverse. Then, for all $T \in VN(D)$ and $u, v \in A(C)$,

$$\langle r^*_{C,D}(u|_D \cdot T), v \rangle = \langle u|_D \cdot T, v|_D \rangle = \langle T, (uv)|_D \rangle$$
$$= \langle r^*_{C,D}(T), uv \rangle = \langle u \cdot r^*_{C,D}(T), v \rangle,$$

and hence, for all $S \in VN_D(C)$ and $u, v \in A(C)$,

$$\langle r^*_{C,D}(u|_D \cdot j_{C,D}(S)), v \rangle = \langle u \cdot r^*_{C,D}(j_{C,D}(S)), v \rangle$$
$$= \langle r^*_{C,D}(j_{C,D}(S)), uv \rangle = \langle S, uv \rangle = \langle u \cdot S, v \rangle$$
$$= \langle r^*_{C,D}(j_{C,D}(u \cdot S)), v \rangle.$$

This implies that

$$u|_D \cdot j_{C,D}(S) = j_{C,D}(u \cdot S)$$

for all $S \in VN_D(C)$ and $u \in A(C)$. Let

$$R = r^*_{G,H} \circ j_{K,H} \circ Q \circ j_{G,K} \circ P : VN(G) \to VN_H(G).$$

Then R is a linear map of norm one and, as is easily checked, a projection onto $VN_H(G)$. Moreover, R is completely positive (respectively, completely contractive) whenever P and Q are completely positive (respectively, completely contractive). Finally, for $T \in VN(G)$ and $u \in A(G)$,

$$\langle v \cdot R(T), u \rangle = \langle R(T), vu \rangle = \langle (j_{K,H} \circ Q \circ j_{G,K} \circ P)(T), (vu)|_H \rangle$$
$$= \langle (Q \circ j_{G,K} \circ P)(T), (vu)|_K \rangle = \langle Q(v|_K \cdot (j_{G,K}(P(T))), u|_K \rangle$$
$$= \langle Q(j_{G,K}(v \cdot P(T))), u|_k \rangle$$
$$= \langle (Q \circ j_{G,K} \circ P)(v \cdot T), u|_K \rangle$$
$$= \langle j_{K,H} \circ Q \circ j_{G,K} \circ P(v \cdot T), u|_H \rangle$$
$$= \langle r^*_{G,H}(j_{K,H} \circ Q \circ j_{G,K} \circ P(v \cdot T)), u \rangle$$
$$= \langle R(v \cdot T), u \rangle.$$

Thus R commutes with the action of $A(G)$. $\square$

If K and L are closed subgroups of G such that $K \subseteq L$, then associated to K are ideals of $A(G)$ and $A(L)$, respectively. For clarity, we denote the latter by $I_L(K)$, that is, $I_L(K) = \{v \in A(L) : v|_K = 0\}$.

In the sequel we let $A_c(G)$ denote the dense linear subspace of $A(G)$ consisting of all compactly supported functions in $A(G)$.

Lemma 3.2. *Let G be an amenable locally compact group and $\mathscr{L}$ a collection of open subgroups of G such that every compact subset of G is contained in some $L\in\mathscr{L}$. Let $c>0$ and H is a closed subgroup of G, and suppose that, for each $L\in\mathscr{L}$, $I_L(H\cap L)$ has an approximate identity with norm bounded by c. Then $I(H)$ has an approximate identity with norm bounded by c.*

Proof. Let $u\in I(H)$ and $\varepsilon>0$. Since G is amenable, there exists $v\in A_c(G)$ such that $||u-uv||\leqslant\varepsilon$. Choose $L\in\mathscr{L}$ such that v vanishes on $G\backslash L$. By hypothesis, there exists $w_1\in I_L(H\cap L)$ such that $||w_1||\leqslant c$ and

$$||(uv)|_L-(uv)|_L w_1||_{A(L)}\leqslant\varepsilon.$$

Let w denote the trivial extension of w_1 to all of G. Then $w\in I(H)$ and $||w||_{A(G)}=||w_1||_{A(L)}$ [5, Proposition 3.21]. Finally, since $\mathrm{supp}\,(uv)\subseteq L$,

$$\begin{aligned}||u-uw||_{A(G)}&\leqslant ||u-uv||_{A(G)}+||uv-uvw||_{A(G)}+||uv-u||_{A(G)}||w||_{A(G)}\\&\leqslant\varepsilon+||(uv)|_L-(uv)|_L w_1||_{A(L)}+\varepsilon||w_1||_{A(L)}\leqslant\varepsilon(2+c).\end{aligned}$$

This shows that $I(H)$ has a bounded approximate identity with bound c. $\square$

Theorem 3.3. *Let G be a locally nilpotent locally compact group and H a closed subgroup of G. Then the ideal $I(H)$ has an approximate identity with norm bounded by 2.*

Proof. By Lemma 3.2 we can assume that G is compactly generated and hence that G is nilpotent. Let

$$\{e\}=Z_0(G)\subseteq Z_1(G)\subseteq\cdots\subseteq Z_m(G)=G$$

denote the ascending central series of G, that is, $Z_{j+1}(G)/Z_j(G)$ equals the centre of $G/Z_j(G)$ for $0\leqslant j\leqslant m-1$. Let $H_j=\overline{HZ_j(G)}$, $0\leqslant j\leqslant m$, so that $H_0=H$ and $H_m=G$. Then H_j is normal in H_{j+1} for each $0\leqslant j\leqslant m-1$. Lemma 3.1 now yields the existence of norm 1 projection from $VN(G)$ onto $VN_H(G)$ which commutes with the action of $A(G)$. Consequently $I(H)$ has an approximate identity with norm bound 2. $\square$

Lemma 3.4. *Let G be a locally compact group and K a compact normal subgroup of G. Let H be a closed subgroup of G such that $H\subseteq G_0C$ for some compact subset C of G. Let $\mathscr{U}$ be a neighbourhood basis of the identity in HK. Then*

$$\bigcap_{U\in\mathscr{U}}\overline{HUH}=H.$$

Proof. For $x \in G$, let $\gamma(x): K \to K$ denote the inner automorphism $y \to x^{-1}yx$, $y \in K$. The homomorphism $\gamma : x \to \gamma(x)$ from G into the automorphism group $\mathrm{Aut}(K)$ of K is continuous [14]. Thus $\gamma(G_0)$ is contained in the connected component of the identity in $\mathrm{Aut}(K)$. Now, by a theorem of Iwasawa (see [14, p. 439]), for any compact group this connected component consists entirely of inner automorphisms. Thus, for each $x \in G_0$, there exists some $k \in K$ such that $\gamma(x) = \gamma(k)$.

We have to show that if $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets in H and $\{u_\alpha\}$ is a net in HK such that $u_\alpha \to e$ and $x_\alpha u_\alpha y_\alpha \to a$ for some $a \in HK$, then $a \in H$. By hypothesis, there exist $c_\alpha \in C$ and $z_\alpha \in G_0$ such that $y_\alpha = z_\alpha c_\alpha$. Since C is compact, after passing to a subnet if necessary, we can assume that $c_\alpha \to c$ for some $c \in C$. By what we have noticed above, for each α there exists $k_\alpha \in K$ such that $\gamma(z_\alpha) = \gamma(k_\alpha)$. Since $u_\alpha \to e$, there exist $v_\alpha \in H$ and $w_\alpha \in K$ such that $u_\alpha = v_\alpha w_\alpha$ and $v_\alpha \to e$. It follows that

$$x_\alpha u_\alpha y_\alpha = (x_\alpha v_\alpha y_\alpha)c_\alpha^{-1}(z_\alpha^{-1} w_\alpha z_\alpha)c_\alpha = (x_\alpha v_\alpha y_\alpha)c_\alpha^{-1}(k_\alpha^{-1} w_\alpha k_\alpha)c_\alpha$$

converges to a. Now, since K is compact, $k_\alpha \in K$ and $w_\alpha \to e$, we conclude that $c_\alpha^{-1}(k_\alpha^{-1} w_\alpha k_\alpha)c_\alpha \to e$ and hence $x_\alpha v_\alpha y_\alpha \to a$, whence $a \in H$, as required. □

Corollary 3.5. *Let G be a locally compact group and K a compact normal subgroup of G. If H is an almost connected closed subgroup of G, then HK has the H-separation property.*

Proof. Since H is almost connected it satisfies the hypothesis of Lemma 3.4. Moreover, HK is also almost connected since K is compact. Now apply Lemma 3.4 and [18, Theorem 1.3]. □

Corollary 3.6. *Let G be an almost connected locally compact group and K a compact normal subgroup of G. Then HK has the H-separation property for any closed subgroup H of G.*

Proof. Since G is almost connected, it is a projective limit of Lie groups G/K_α. In particular, the subgroup HKK_α/K_α of G/K_α is locally connected. It follows from Lemma 3.4 and [18, Theorem 1.3] that HKK_α/K_α has the HK_α/K_α-separation property. Since $H = \cap_\alpha HK_\alpha$, we readily conclude that HK has the H-separation property. □

Theorem 3.7. *Let G be an almost connected group with a compact normal subgroup K, and suppose that, for every closed subgroup L of G containing K, there is a norm one projection from $VN(G)$ onto $VN_L(G)$ commuting with the action of $A(G)$. Then, for each closed subgroup H of G, $I(H)$ has an approximate identity with norm bounded by 2. The conclusion in particular holds if G/K is nilpotent.*

Proof. Let H be any closed subgroup of G. It suffices to show that there is a norm one projection from $VN(G)$ onto $VN_H(G)$ commuting with the action of $A(G)$. By Corollary 3.6 and the hypothesis, there exist norm one projections from $VN(HK)$

onto $VN_H(HK)$ and from $VN(G)$ onto $VN_{HK}(G)$ commuting with the action of $A(HK)$ and of $A(G)$, respectively. Now apply Lemma 3.1.

When G/K is nilpotent, the hypothesis of the theorem is satisfied. Indeed, this follows as the in the proof of Theorem 3.3 by considering, with $q: G \to G/K$ denoting the quotient homomorphism, the ascending sequence of closed subgroups

$$L = Lq^{-1}(Z_0(G/K)) \subseteq \overline{Lq^{-1}(Z_1(G/K))} \subseteq \ldots \subseteq \overline{Lq^{-1}(Z_m(G/K))}$$

and noticing that $Lq^{-1}(Z_{j-1}(G/K))$ is normal in $Lq^{-1}(Z_j(G/K))$. $\square$

Theorem 3.7, in particular, includes all connected groups with relatively compact commutator subgroups. In general, such groups are neither nilpotent nor SIN-groups. To construct examples, simply take a non-solvable compact connected group K and a one-parameter group of topological automorphisms of K, and form the resulting semidirect product of $\mathbb{R}$ with K. More specifically, let for instance $K = SO(n), n \geqslant 3$, fix a two-dimensional linear subspace V of $\mathbb{R}^n$ and, for $t \in \mathbb{R}$, let $D_t \in SO(n)$ act on V by rotation with angle t and on the orthogonal complement of V as the identity. Then form the semidirect product $G = \mathbb{R} \ltimes SO(n)$ given by conjugation with D_t on $SO(n)$.

Lemma 3.8. *Let G be a projective limit of groups $G_\alpha = G/K_\alpha$. Let E be a closed subset of G such that E is a spectral set. Suppose that, for some $c > 0$, the ideal $I(EK_\alpha/K_\alpha)$ of $A(G_\alpha)$ has an approximate identity of norm $\leqslant c$ for every α. Then $I(E)$ has an approximate identity of norm $\leqslant c$.*

Proof. We have to show that given $u \in I(E)$ and $\varepsilon > 0$, there exists $v \in I(E)$ such that $||v|| \leqslant c$ and $||u - uv|| < \varepsilon$. Since E is a spectral set, the ideal

$$J(E) = \{w \in A_c(G) : w \text{ vanishes on a neighbourhood of } E\}$$

is dense in $I(E)$. Thus we can assume that $u \in J(E)$. Let $u(x) = \langle \lambda(x)\xi, \eta \rangle$, where $\xi, \eta \in L^2(G)$, and choose $\delta > 0$ such that $\delta(1 + (1 + c)||\eta||) \leqslant \varepsilon$. Since $\operatorname{supp} u$ is compact and disjoint from E, there exists a neighbourhood V of the identity such that $EV \cap \operatorname{supp} u = \emptyset$ and $||L_y\xi - \xi||_2 \leqslant \delta$ for all $y \in V$.

Choose α such that $K_\alpha \subseteq V$, set $K = K_\alpha$ and normalize the Haar measure on K. Then $u \in I(EK)$. Define $\tilde{u}$ on G/K by $\tilde{u}(xK) = \int_K u(xk)\, dk$. Then $\tilde{u} \in A(G/K)$ and hence $\tilde{u} \in I(EK/K)$ since u vanishes on EK. By hypothesis, there exists $w \in I(EK/K)$ so that $||w||_{A(G/K)} \leqslant c$ and $||\tilde{u}w - \tilde{u}||_{A(G/K)} \leqslant \delta$. Let $v \in A(G)$ be defined by $v(x) = w(xK),\ x \in G$. Then $v \in I(E)$ and $||v|| = ||w||_{A(G/K)}$. Now, with $q: G \to G/K$ denoting the quotient homomorphism, for $x \in G$,

$$u(x) - (\tilde{u} \circ q)(x) = \langle \lambda(x)\xi, \eta \rangle - \int_K \langle \lambda(x)L_k\xi, \eta \rangle\, dk.$$

Since $||\xi - L_k\xi|| \leqslant \delta$ for all $k \in K$, this implies that

$$||u - \tilde{u} \circ q|| \leqslant ||\eta||_2 \left|\left| \xi - \int_K L_k \xi \, dk \right|\right|_2 \leqslant ||\eta||_2 \sup_{k \in K} ||\xi - L_k\xi||_2 \leqslant \delta ||\eta||_2.$$

Finally, it follows that

$$\begin{aligned} ||u - uv|| &\leqslant ||u - \tilde{u} \circ q|| + ||\tilde{u} \circ q - (\tilde{u} \circ q)v|| + ||(\tilde{u} \circ q)v - uv|| \\ &\leqslant (1 + ||v||)||u - \tilde{u} \circ q|| + ||\tilde{u} \circ q - (\tilde{u} \circ q)(w \circ q)|| \\ &\leqslant (1 + c)||u - \tilde{u} \circ q|| + ||\tilde{u} - \tilde{u}w||_{A(G/K)} \leqslant (1 + c)\delta||\eta|| + \delta \leqslant \varepsilon. \qquad \square \end{aligned}$$

We continue this section with two examples. It follows from Lemma 3.2 that if G is an amenable group which can be written as the union of a net of ascending open subgroups each of which is an SIN-groups, then $I(H)$ has an approximate identity with norm bounded by 2 for every closed subgroup H of G. There are plenty of such groups that are not SIN-groups themselves. Our first example presents just one class of such groups.

Similarly, if G is a projective limit of (locally) nilpotent groups then, by Theorem 3.3 and Lemma 3.8, $I(H)$ has an approximate identity with norm bounded by 2 for every closed subgroup H of G. The second example shows that such a group need not be locally nilpotent.

Example 3.9. For $n \geqslant 3$, let Γ_n denote the multiplicative group of real $(n \times n)$-matrices

$$A = [a_{jk}] \text{ with } a_{jj} = 1 \text{ and } a_{jk} = 0 \text{ for } 1 \leqslant j \leqslant n, \ 1 \leqslant k < j.$$

Define subgroups Λ_n and Δ_n of Γ_n by

$$\Lambda_n = \{A \in \Gamma_n : a_{jj+1} = 0 \text{ for } 1 \leqslant j \leqslant n-1\}$$

and

$$\Delta_n = \{A \in \Lambda_n : a_{jk} \in \mathbb{Z} \text{ for } 1 \leqslant j \leqslant k-2, \ k \leqslant n\}.$$

Let $H_n = \Gamma_n/\Delta_n$ and $K_n = \Lambda_n/\Delta_n$, and form the algebraic direct products

$$G = \prod_{n=3}^{\infty} H_n \quad \text{and} \quad K = \prod_{n=3}^{\infty} K_n.$$

Endow each K_n with its natural compact topology and K with the product topology, and define a group topology on G by declaring K to be open. Then G is the projective limit of the groups $G_m = G/\prod_{j=m}^{\infty} K_j$, $j \geqslant 3$. Since K_n is $(n-2)$-step nilpotent, K fails to be nilpotent, and hence G is not locally nilpotent.

Example 3.10. For $j \in \mathbb{N}$, let D_j be a discrete group and F_j a finite group of automorphisms of D_j. Let $K = \prod_{j=1}^{\infty} F_j$, the direct product with the product topology, and let N be the direct sum of all D_j, a discrete group. Form $G = K \ltimes N$, the semidirect product of K and N, where elements of K act on N componentwise. It is obvious that G fails to be an SIN-group whenever F_j is nontrivial for infinitely many j. For $n \in \mathbb{N}$, let $\mathbb{N}_n$ be the direct sum of $D_1, \ldots, D_n$ and let $G_n = K \ltimes N_n$. Then $G = \bigcup_{n=1}^{\infty} G_n$, each G_n is open in G, and

$$G_n = \left(\left(\prod_{j=1}^{n} F_j \right) \ltimes N_n \right) \times \left(\prod_{j=n+1}^{\infty} F_j \right),$$

so that G_n is a direct product of a discrete group and a compact group. In particular, G_n is an SIN-group.

We have already seen that in many cases there exists an invariant projection from $VN(G)$ onto $VN_H(G)$. We conclude this section by showing that for several classes of locally compact groups G, for any closed subgroup H of G there even exists a completely positive invariant projection of norm 1 from $VN(G)$ onto $VN_H(G)$. For this, the following lemma is needed.

Lemma 3.11. *Let G be a locally compact group and let H be a closed subgroup of G. If G has the H-separation property, then there exists a completely positive invariant projection P from $VN(G)$ onto $VN_H(G)$ such that $P(I) = I$.*

Proof. For $u \in P(G)$, define $\Lambda_u \in \mathscr{B}(VN(G))$ by $\Lambda_u(T) = u \cdot T$. It is known that Λ_u is completely positive [1, Proposition 4.2].

Let $\mathscr{K}_H = \overline{\{\Lambda_u : u \in P_H(G)\}}$ where the closure is taken in the weak*-operator topology (i.e. a net $\{\Lambda_\alpha\}_\alpha$ in $\mathscr{B}(VN(G))$ converges to $\Lambda \in \mathscr{B}(VN(G))$ if and only $\Lambda_\alpha(T) \to \Lambda(T)$ in the weak*-topology of $VN(G)$ for every $T \in VN(G)$. Then $\mathscr{K}_H$ is convex and compact, and each $\Lambda \in \mathscr{K}_H$ is completely positive (see [23, Theorem 6.4]). Now, an application of the Markov–Kakutani fixed point theorem as in the proof of [16, Proposition 3.1] shows that there exists $P \in \mathscr{K}_H$ such that $u \cdot P(T) = P(T)$ for all $u \in P_H(G)$ and $T \in VN(G)$. Then P is a projection from $VN(G)$ onto $VN_H(G)$ (see the proof of [16, Proposition 3.1]). Finally, since each Λ_u commutes with the action of $A(G)$, so does P. □

Theorem 3.12. *Let G be a locally compact group which satisfies one of the following three conditions:*

(i) *G is an SIN-group.*
(ii) *G is nilpotent.*
(iii) *G is almost connected and contains a compact normal subgroup K such that G/K is nilpotent.*

Let H be any closed subgroup of G. Then there exists a completely positive invariant projection from $VN(G)$ onto $VN_H(G)$ such that $P(I) = I$.

Proof. The proof follows by applying Lemma 3.11 together with Lemma 3.1 and Corollary 3.6 as in the proofs of Theorems 3.3 and 3.7. □

4. The algebras $A_p(G)$

Let G be a locally compact group. Herz has introduced L^p-versions of the Fourier algebra for all $1<p<\infty$. He defined $A_p(G)$ to be the space of all functions u of the form $u(x)=\sum_{n=1}^{\infty}(f_n*\tilde{g}_n)(x^{-1})$, where $f_n\in L^p(G), g_n\in L^q(G), \frac{1}{p}+\frac{1}{q}=1$, and $\sum_{n=1}^{\infty}||f_n||_p||g_n||_q<\infty$. With the norm

$$||u||_{A_p(G)}=\inf\left\{\sum_{n=1}^{\infty}||f_n||_p||g_n||_q : u(x)=\sum_{n=1}^{\infty}(f_n*\tilde{g}_n)(x^{-1})\right\}$$

and pointwise multiplication $A_p(G)$ becomes a commutative Banach algebra. Of course, $A_2(G)=A(G)$. The algebras $A_p(G)$ are nowadays usually referred to as the Herz–Figà–Talamanca algebras.

In general, for $p\neq 2$, these algebras share many common characteristics with the Fourier algebra. For example, the existence of a bounded approximate identity in $A_p(G)$ characterizes amenability for the group G. However, if $p\neq 2$, $A_p(G)$ is no longer predual of a von Neumann algebra. This means that the powerful theory of operator algebras is not available and hence to attack analogous problems for $A_p(G)$, $p\neq 2$, methods different from those that have proved successful for studying $A(G)$ are often needed. In particular, there is currently no satisfactory operator space structure on $A_p(G)$ that would allow us to extend Ruan's result to this class of algebras. Nonetheless, we will still be able to use our results on $A(G)$ that we obtained by exploiting its operator space structure to gain some analogous results for the more general algebras $A_p(G)$.

Let G be an amenable group and $1<p<\infty$. Then, by a result of Herz [12, Theorem C], $A(G)=A_2(G)\subseteq A_p(G)$ and $||u||_{A_p(G)}\leqslant ||u||_{A(G)}$. For a closed subset E of G, let

$$I_p(E)=\{u\in A_p(G) : u(x)=0 \text{ for all } x\in E\}.$$

Lemma 4.1. *Let G be an amenable locally compact group. Let E be a closed subset of G that is of spectral synthesis for $A_p(G)$. If $I_2(E)$ has a bounded approximate identity with norm bounded by c, then so does $I_p(E)$.*

Proof. Let $\Gamma : A_2(G)\to A_p(G)$ be the above injection. It is clear that $\Gamma(I_2(E))\subseteq I_p(E)$. In fact, we will show that $\Gamma(I_2(E))=I_2(E)$ is dense in $I_p(G)$ with respect to the norm of $A_p(G)$. To see this, we first choose a compact subset K in G with $K\cap E=\emptyset$ and an open set U in G with $K\subseteq U\subseteq G\backslash E$. Let $u\in A_p(G)$ be such that $\operatorname{supp} u\subseteq K$. We can choose a sequence $\{v_n\}$ in $A(G)$ such that $v_n \to u$ in $A_p(G)$. However, we can also find a $v \in A(G)$ such that $v|_K = 1$ and $\operatorname{supp} v \subseteq U$. It follows that $v_n v \in I_2(E)$ and

$$\lim_{n\to\infty} ||u - v_n v||_{A_p(G)} = \lim_{n\to\infty} ||uv - v_n v||_{A_p(G)}$$
$$\leqslant ||v||_{A_p(G)} \lim_{n\to\infty} ||u - v_n||_{A_p(G)} = 0.$$

This shows that we can approximate u by functions in $\Gamma(I_2(E))$. However, since E is a set of synthesis for $A_p(G)$, such functions u are dense in $I_p(G)$. Thus $\Gamma(I_2(E))$ is dense in $I_p(E)$.

Assume that $\{v_\alpha\}$ is an approximate identity in $I_2(E)$ such that $||v_\alpha||_{A_2(G)} \leqslant$ for all α. Then, since Γ is a contraction, $||\Gamma(v_\alpha)|| \leqslant c$ for all α. If $v \in \Gamma(I_2(E))$, then

$$\lim_{\alpha} ||\Gamma(v) - \Gamma(v_\alpha)||_{A_p(G)} \leqslant \lim_{\alpha} ||v - v_\alpha||_{A_2(G)} = 0.$$

Since $\Gamma(I_2(E))$ is dense in $I_p(E)$, it follows that $\{\Gamma(v_\alpha)\}_\alpha$ is the desired approximate identity. □

We can now extend Theorem 1.3 to the algebras $A_p(G)$.

Corollary 4.2. *Let G be an amenable locally compact group and let H be a closed subgroup of G. Let $1 < p < \infty$. Then the ideal $I_p(H)$ has a bounded approximate identity.*

Proof. We note that if G is amenable, then every closed subgroup of G is a set of synthesis for $A_p(G)$ [13, Propositions 1 and 2]. It now follows immediately from Theorem 1.3 and from Lemma 4.1 that $I_p(H)$ has a bounded approximate identity. □

The conclusion of the preceding corollary has recently been verified when H is a so-called neutral subgroup of a locally compact group G [3, Theorem 6].

In [9, Theorem 3.7] it was shown that if G is an amenable SIN-group and if $E \in \mathscr{R}_c(G)$, then for each $1 < p < \infty$, $I_p(E)$ has a bounded approximate identity. It is desirable to extend this result to all amenable groups. To that end, observe that the p-analogue of [11, Theorem 3.8] can readily been shown to hold for all $1 < p < \infty$. It is then easy to verify that Lemma 2.1 holds in this more general context. With this lemma, the proof of Lemma 2.2 also holds for $1 < p < \infty$. We can now appeal to Corollary 4.2 to conclude:

Theorem 4.3. *Let G be an amenable locally compact group and let $E \in \mathscr{R}_c(G)$. Then, for each $1 < p < \infty$, E is a set of synthesis for $A_p(G)$ and $I_p(E)$ has a bounded approximate identity.*

Acknowledgments

The authors are grateful to the referee for pointing out some misprints and for providing Refs. [3,4].

References

[1] J. De Canniere, U. Haagerup, Multipliers of the Fourier algebra of some semisimple Lie groups and their discrete subgroups, Amer. J. Math. 107 (1985) 455–500.
[2] J. Delaporte, A. Derighetti, Best bounds for the approximate units of certain ideals of $L^1(G)$ and $A_p(G)$, Proc. Amer. Math. Soc. 124 (1996) 1159–1169.
[3] J. Delaporte, A. Derighetti, Invariant projections and convolution operators, Proc. Amer. Math. Soc. 129 (2001) 1427–1435.
[4] A. Derighetti, Relations entre les convoluteurs d'un groupe localement compact et ceux d'un sous-groupe fermé, Bull. Sci. Math. 106 (1982) 69–84.
[5] E. Effros, Z.-J. Ruan, Operator Spaces, Oxford University Press, Oxford, 2000.
[6] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964) 181–236.
[7] B. Forrest, Amenability and bounded approximate identities in ideals of $A(G)$, Illinois J. Math. 34 (1990) 1–25.
[8] B. Forrest, Amenability and ideals in $A(G)$, J. Austral. Math. Soc. Ser. A 53 (1992) 143–155.
[9] B. Forrest, Amenability and the structure of the algebras $A_p(G)$, Trans. Amer. Math. Soc. 343 (1994) 233–243.
[10] B. Forrest, P. Wood, Cohomology and the operator space structure of the Fourier algebra and its second dual, Indiana Univ. Math. J. 50 (2001) 1217–1240.
[11] J.E. Gilbert, On projections of $L^\infty(G)$ onto translation-invariant subspaces, Proc. London Math. Soc. 19 (1969) 69–88.
[12] C. Herz, The theory of p-spaces with an application to convolution operators, Trans. Amer. Math. Soc. 154 (1971) 69–82.
[13] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1973) 91–123.
[14] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis. I, Springer, Berlin, 1963.
[15] B. Host, Le théorème des idempotents dans $B(G)$, Bull. Soc. Math. France 114 (1986) 215–223.
[16] E. Kaniuth, A.T. Lau, A separation property of positive definite functions on locally compact groups and applications to Fourier algebras, J. Funct. Anal. 175 (2000) 89–110.
[17] E. Kaniuth, A.T. Lau, Spectral synthesis for $A(G)$ and subspaces of $VN(G)$, Proc. Amer. Math. Soc. 129 (2001) 3253–3263.
[18] E. Kaniuth, A.T. Lau, On a separation property of positive definite functions on locally compact groups, Math. Z., to appear.
[19] A.T. Lau, V. Losert, Weak*-closed complemented invariant subspaces of $L^\infty(G)$ and amenable locally compact groups, Pacific J. Math. 123 (1986) 149–159.
[20] H. Leptin, Sur l'algèbre de Fourier d'un groupe localement compact, C.R. Acad. Sci. Paris Ser. A 266 (1968) 1180–1182.
[21] T.S. Liu, A. van Rooij, J. Wang, Projections and approximate identities for ideals in group algebras, Trans. Amer. Math. Soc. 175 (1973) 469–482.
[22] N. Lohoué, Sur les convoluteurs de $L^p(G)$, C.R. Acad. Sci. Paris 278 (1974) 1543–1545.
[23] V. Paulsen, Completely Bounded Maps and Dilations, in: Research Notes in Mathematics, Vol. 146, Pitman, London, 1986.
[24] Z.-J. Ruan, The operator amenability of $A(G)$, Amer. J. Math. 117 (1995) 1449–1476.
[25] B.M. Schreiber, On the coset ring and strong Ditkin sets, Pacific J. Math. 32 (1970) 805–812.

[26] M. Takesaki, N. Tatsuuma, Duality and subgroups. II, J. Funct. Anal. 11 (1972) 184–190.
[27] C.R. Warner, A class of spectral sets, Proc. Amer. Math. Soc. 57 (1976) 99–102.
[28] P.J. Wood, Complemented ideals in the Fourier algebra of a locally compact group, Proc. Amer. Math. Soc. 128 (2000) 445–451.