BEST BOUNDS FOR APPROXIMATE IDENTITIES IN IDEALS OF THE FOURIER ALGEBRA VANISHING ON SUBGROUPS

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Abstract. In this paper we show that if G is an amenable locally compact group and if H is a closed subgroup then the ideal $I(H)$ has an approximate identity of norm 2. If H is not open this bound is best posible.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure. If $1 < p < \infty$, we let $A_p(G)$ be the subspace of $C_0(G)$ consisting of functions of the form

$$u(x) = \sum_{i=1}^{\infty} f_i * g_i^{\vee}$$

where $g_i \in L^p(G)$, $f_i \in L^q(G)$, $\frac{1}{p} + \frac{1}{q} = 1, g_i^{\vee}(x) = g_i(x^{-1})$ and

$$\sum_{i=1}^{\infty} \| g_i \|_p \| f_i \|_q < \infty.$$

Moreover, we let

$$\| u \|_{A_p(G)} = \inf\{\sum_{i=1}^{\infty} \| g_i \|_p \| f_i \|_q | \; u(x) = \sum_{i=1}^{\infty} f_i * g_i^{\vee}\}.$$

With respect to the norm $\| u \|_{A_p(G)}$ and pointwise operations, $A_p(G)$ becomes a commutative Banach algebra called the *Herz-Figà-Talamanca $p-$algebra of G*. In the case $p = 2$, we simply write $A(G)$ for $A_2(G)$. $A(G)$ is called the *Fourier algebra of G*. It was introduced for noncommutive groups by Eymard in [4]. The dual of $A(G)$ is the von Neumann

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algebra $VN(G)$ generated by the left translation operators acting on the Hilbert space $L^2(G)$.

Until recently, one of the most interesting open problems in the ideal theory of $A(G)$ was to determine which closed ideals had bounded approximate identities. This problem was solved by the authors together with E. Kaniuth and A.T. Lau in [6] where it was shown that a closed ideal I has a bounded approximate identity if and only I is the set of all functions in $A(G)$ vanishing on a set E where E is in the closed coset ring of G. One of the key steps in this process was to show that if G is an amenable group and if H is a closed subgroup of G, then the ideal $I(H) = \{u \in A(G) \mid u(h) = 0 \text{ for all } h \in H\}$ always admits a bounded approximate identity. This result was obtained by using techniques from cohomology and from operator spaces. It improved upon earlier work of the first author who showed in [5] that if G is an amenable $[SIN]-$group and if H is a closed subgroup of G, then the ideal $I(H)$ always admits a bounded approximate identity by showing that the pair (G, H) satisfy a strong separation property, namely that for every $x \in G, x \notin H$ there exists a continuous positive definite function φ such that $\varphi(x) \neq 1$ but $\varphi(h) = 1$ for each $h \in H$. This property has been studied extensively by Kaniuth and Lau in [9] and [10] who showed that it is of substantial interest on its own. In particular, they showed that while it is possible to establish the separation property for various types of subgroups, the general property fails outside of the class of $[SIN]-$groups. However, in [9], they were also able to show that if G is an amenable locally compact group and if G satisfies the H-separation property then there exists a norm-1 projection P from $VN(G)$ onto $VN_H(G) = I(H)^{\perp}$ that commutes with the module action of $A(G)$ on $VN(G)$. From this they were able to appeal to Proposition 1 of [2] to conclude that $I(H)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \Omega}$ where $\| u_\alpha \| = 2$ for each α. The method in [5] gave an approximate identity for $I(H)$ of norm at most 3 when $G \in [SIN]$. This is significant since, using ideas from [3], Kaniuth and Lau were able to show that if H is any closed and nonopen subgroup of a locally compact group G for which $I(H)$ has a bounded approximate identity, then 2 is the best norm possible bound for the approximate identity [9, Theorem 3.3]. Delaporte and Derighetti established the same result independently in [3, Theorem 10] under the additional assumption that G is amenable.Their work involved subtle estimates concerning the norms of various convolution operators. In contrast, it is always the case that if G is amenable, then $A(G)$ has a norm-1 bounded approximate identity. In [6], we were able to build on the earlier work of Kaniuth and

Lau to extend the class of subgroups known to have approximate identities with optimal norm bounds. However, our method that allowed us to establish the existence of bounded approximate identities in $I(H)$ whenever G is amenable was not sufficiently refined to provide the desired norm bound for all such H. In this note, we will again exploit the operator amenability of $A(G)$ to give an entirely different proof that if G is amenable, then $I(H)^{\perp}$ is invariantly complemented via a norm-1 projection, and in so doing establish the desired optimal norm bound.

2. The Main Theorem

A *completely contractive Banach algebra* is a Banach algebra $\mathcal{A}$ that is also an operator space for which the multiplication map

$$m : \mathcal{A}\widehat{\otimes}_{op}\mathcal{A} \ \to \mathcal{A}$$

is completely contractive where $\mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$ denotes the operator space projective tensor product. This definition appeared first in [12]. The suitable class of modules for such an algebra are the *completely contractive* $\mathcal{A}-$*modules*. That is, those modules that are also operator spaces for which the module multiplication maps are again completely contractive. If $\mathcal{X}$ is a completely contractive (left/right) $\mathcal{A}-$module, then its dual space $\mathcal{X}^*$ is also a completely contractive (right/left) $\mathcal{A}-$module with respect to the usual adjoint module action. If $\mathcal{X}, \mathcal{Y}$ are completely contractive left $\mathcal{A}-$modules, then the space of completely bounded linear maps $CB(\mathcal{X}, \mathcal{Y})$ is a completely contractive $\mathcal{A}-$module via

$$(a \cdot T)(x) = a \cdot (T(x)) \text{ and } (T \cdot a)(x) = T(a \cdot x)$$

for $a \in \mathcal{A}, T \in CB(\mathcal{X}, \mathcal{Y})$ and $x \in \mathcal{X}$. This leads to a complete contraction

$$\Theta = \Theta_{\mathcal{X},\mathcal{Y}} : \mathcal{A}\widehat{\otimes}_{op}\mathcal{A} \to CB(CB(\mathcal{X}, \mathcal{Y}))$$

given by

$$\Theta(a \otimes b)T = a \cdot T \cdot b.$$

We note that

$$\Theta(u \cdot a)T = [\Theta(u)T] \cdot a \text{ and } \Theta(a \cdot u)T = a \cdot [\Theta(u)T]$$

for each $u \in \mathcal{A}\widehat{\otimes}_{op}\mathcal{A}, a \in \mathcal{A}$ and $T \in CB(\mathcal{X}, \mathcal{Y})$, where $u \cdot a = b \otimes ca$ and $a \cdot u = (ab) \otimes c$ when $u = b \otimes c \in \mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$.

Following Johnson [8], we say that a completely contractive Banach algebra $\mathcal{A}$ is *operator* $d-$*amenable* if there is an approximate diagonal in $\mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$ of norm bounded by d. That is, if there esists a net $\{u_\alpha\}$ in $\mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$ such that

$$\begin{array}{ll} i) & \lim_{\alpha} \| a\cdot u_\alpha - u_\alpha \cdot a \| = 0 \\ ii) & \{m(u_\alpha)\} \text{ is an approximate identity for } \mathcal{A} \\ iii) & \sup_{\alpha} \| u_\alpha \| \leq d \end{array}$$

The following result is an adaptation to the operator space setting of [13, Theorem 2.3.13]. It improves on [14, Theorem 3] in the sense that we gain better control on the norm of the splitting map.

2.1. **Lemma.** *Suppose that $\mathcal{A}$ is an operator $d-$amenable completly contractive Banach algebra, $\mathcal{X}$ is an essential completely contractive right $\mathcal{A}-$module and $\mathcal{Y}$ is a weak*-closed left $\mathcal{A}-$submodule of $\mathcal{X}^*$ that is completely complemented via a completely contractive projection $Q : \mathcal{X}^* \rightarrow \mathcal{Y}$. Then there exists a projection $P : \mathcal{X}^* \rightarrow \mathcal{Y}$ which is also an $\mathcal{A}-$module map with $\| P \|_{cb} \leq d$.*

Proof. Let $\Theta = \Theta_{\mathcal{X}^*,\mathcal{Y}}$ be as above. Let $\{u_\alpha\}$ be a $d-$bounded approximate diagonal in $\mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$ and let $\mathcal{U}$ be an ultrafilter on the index set of $\{u_\alpha\}$ which dominates the order filter. We let

$$P = point - weak^* \lim_{\alpha \in U \in \mathcal{U}} \Theta(u_\alpha)Q.$$

That is for each $f \in \mathcal{X}^*$ and $x \in \mathcal{X}$, we have

$$< Pf, x >= \lim_{\alpha \in U \in \mathcal{U}} < [\Theta(u_\alpha)Q]f, x > .$$

It is then clear that $\| P \|_{cb} \leq \sup_{\alpha} \| u_\alpha \| \| Q \| \leq d$. It is also clear that $[\Theta(u_\alpha)Q]f \in \mathcal{Y}$ whenever $u \in \mathcal{A}\widehat{\otimes}_{op}\mathcal{A}$ and $f \in \mathcal{X}^*$. Since $\mathcal{Y}$ is weak-* closed we get that $Pf \in \mathcal{Y}$ whenever $f \in \mathcal{X}^*$. Furthermore, if $f \in \mathcal{Y}$, then

$$[\Theta(u)Q]f = m(u)\cdot f.$$

To see this let $u = a \otimes b$. Then

$$[\Theta(u)Q]f = a \cdot Q \cdot b(f) = a \cdot (b \cdot f) = m(a \otimes b) \cdot f.$$

Therefore, if $f \in \mathcal{Y}$, then

$$Pf = weak^* \lim_{\alpha \in U \in \mathcal{U}} [\Theta(u_\alpha)Q]f = weak^* \lim_{\alpha \in U \in \mathcal{U}} m(u_\alpha) \cdot f = f.$$

Finally, to see that P is an $\mathcal{A}-$module map, we note that for $a \in \mathcal{A}$ and $f \in \mathcal{X}^*$, we have

$$\begin{aligned} P(a \cdot f) &= weak^* \lim_{\alpha \in U \in \mathcal{U}} [\Theta(u_\alpha \cdot a)Q]f \\ &= weak^* \lim_{\alpha \in U \in \mathcal{U}} [\Theta(a \cdot u_\alpha)Q]f \\ &= a \cdot Pf. \end{aligned}$$

□

As the predual of a von Neumann algebra, $A(G)$ inherits a natural operator space structure [1] with respect to which $A(G)$ becomes a completely contractive Banach algerba. Moreover, $A(G)$ is operator $1-$amenable if and only if G is amenable as a locally compact group [12]. We are now in a position to apply Lemma 2.1 to our problem concerning ideals in $A(G)$.

2.2. **Theorem.** *Let G be an amenable locally compact group. Let H be a closed subgroup of G. Then there exists a unital completely positive projection P from $VN(G)$ onto $VN_H(G) = I(H)^\perp$ such that $P(u \cdot T) = u \cdot P\ (T)$ for all $u \in A(G), T \in VN(G)$.*

Proof. We let $\mathcal{X} = A(G)$, $\mathcal{X}^* = VN(G)$, $\mathcal{Y} = VN_H(G) = I(H)^\perp$. It follows from Proposition 1.2 of [6] that there exists a projection $Q : VN(G) \to I(H)^\perp$ with $\| Q \|_{cb} = \| Q(I) \| = \| I \| = 1$. Since G is amenable, $A(G)$ is operator $1-$amenable [12]. Lemma 2.1 shows that there exists a completely contractive projection $P : VN(G) \to I(H)^\perp$ that commutes with the module action of $A(G)$ on $VN(G)$. Since $P(I) = I$, P is also completely postive.

□

2.3. **Corollary.** *Let G be an amenable locally compact group. Let H be a closed subgroup of G. Then $I(H)$ has an approximate identity with norm bounded by 2. Moreover, if H is nonopen, then 2 is the best possible norm bound for the approximate identity.*

Proof. By Theorem 2.2, there is a completely contractive invariant projection P from $VN(G)$ onto $VN_H(G) = I(H)^\perp$. We can now apply Proposition 11 of [2] to conclude that $I(H)$ has an approximate identity of norm bounded by 2. The last statement follows from [9, Theorem 3.3] or [3, Theorem 10].

□

2.4. **Corollary.** *Let G be an amenable locally compact group. Let H be a closed subgroup of G. Then $I(H)$ is $4-$operator amenable. If H is nonopen, then $I(H)$ is not $d-$operator amenable for any $d < 2$.*

Proof. Let $\{u_\alpha\}_{\alpha\in A}$ be a contractive approximate diagonal in $A(G)\widehat{\otimes}_{op}A(G)$ and let $\{v_\beta\}_{\beta\in B}$ be a $2-$bounded approximate identity in $I(H)$. Let $\Lambda = A \times B^A$ be the product directed set. If $\lambda = (\alpha, (\beta_{\alpha'})_{\alpha'\in A}) \in \Lambda$, we let

$$w_\lambda = u_\alpha \cdot (v_{\beta_\alpha} \otimes v_{\beta_\alpha}).$$

where $I(H)\widehat{\otimes}_{op}I(H)$ is a completely contractive $A(G)\widehat{\otimes}_{op}A(G)$-module in the usual way. Then for each $\lambda \in \Lambda$,

$$\| w_\lambda \|_{I(H)\widehat{\otimes}_{op}I(H)} \leq \| u_\alpha \|_{A(G)\widehat{\otimes}_{op}A(G)} \| v_{\beta_\alpha} \|^2 \leq 4.$$

Moreover, $\{w_\lambda\}_{\lambda\in\Lambda}$ is an approximate diagonal in $I(H)\widehat{\otimes}_{op}I(H)$ since if $v \in I(H)$, then

$$\lim_\lambda m(w_\lambda)v = \lim_\beta \lim_\alpha v_{\beta_\alpha}^2 m(u_\alpha)v = v$$

where the iterated limit identity is as in [11], and

$$\begin{aligned} \lim_\lambda \quad \| \quad & v \cdot w_\lambda - w_\lambda \cdot v \|_{I(H)\widehat{\otimes}_{op}I(H)} \\ = \quad & \lim_{\lambda=(\alpha,(\beta_{\alpha'})_{\alpha'\in A})\in\Lambda} \| (v \cdot u_\alpha) \cdot (v_{\beta_\alpha} \otimes v_{\beta_\alpha}) - (u_\alpha \cdot v) \cdot (v_{\beta_\alpha} \otimes v_{\beta_\alpha}) \|_{I(H)\widehat{\otimes}_{op}I(H)} \\ \leq \quad & \lim_\beta \lim_\alpha \| v \cdot u_\alpha - u_\alpha \cdot v \|_{A(G)\widehat{\otimes}_{op}A(G)} \| v_{\beta_\alpha} \|^2 \\ = \quad & 0. \end{aligned}$$

The final statement follows from [9, Theorem 3.3] or [3, Theorem 10], since if $I(H)$ has a $d-$bounded approximate diagonal, then $I(H)$ has a $d-$bounded approxiamte identity.

□

At this point it does not seem obvious that we can always do better than $4-$amenability for the ideal $I(H)$. Indeed, we do not know if this can be done even for the augmentation ideal $I\{e\}$.

2.5. **Remark.** It is worth noting that the operator space techniques used above for $A(G)$ still allow us to establish the existence of bounded approximate identities in ideals $I_p(H)$ of the Herz-Figa-Talamanca algebras $A_p(G)$ vanishing on closed subgroups of amenable groups when $p \neq 2$. The next theorem is an improvement on [6, Corollary 4.2] (also see [3]). At this point we do not know if the norm bound that we have obtained is optimal when $p \neq 2$.

2.6. **Theorem.** *Let G be an amenable locally compact group. Let H be a closed subgroup of G. Let $1 < p < \infty$. Then there exists a bounded approximate identity $\{u_\alpha\}_{\alpha \in J}$ in $I_p(H)$ with $\| u_\alpha \|_{A_p(G)} \leq 2$ for each $\alpha \in J$.*

Proof. Observe that [6, Corollary 4.2] shows that $I_p(H)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in J}$. Moreover, since G is amenable, the canonical injection $\Gamma : A_2(G) \to A_p(G)$ is a contraction. Given Corollary 2.3, [6, Lemma 4.1] shows that we may also assume that $\| u_\alpha \|_{A_p(G)} \leq 2$ for each $\alpha \in J$.

□

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