PMATH 950, Fall 2018

Assignment #2 Due: November 6.

In all questions below, we assume that G is a second countable locally compact group.

If G is abelian, a *special summability kernel* is a summability kernel $(e_n)_{n=1}^{\infty}$ [i.e. a bounded sequence in $L^1(G)$ for which $\lim_{n\to\infty}\int_G e_n = 1$, and $\lim_{n\to\infty}\int_{G\setminus U}|e_n| = 0$ for any neighbourhood U of e], which additionally satisfies, for each n:

- $0 \leq \widehat{e_n} \leq \widehat{e_{n+1}}$ in $\mathcal{C}_0(\widehat{G})$, and
- $\widetilde{e_n}(x) = \int_{\widehat{G}} \widehat{e_n}(\sigma)\sigma(x)\,d\sigma$ for x in G (Inversion holds).

1. Show that $L^2(G)$ is separable.

2. (a) Let G be abelian, and H an open compact subgroup. Show that $\widehat{1_H} = m(H)1_{H^A}$ where $H^A = \{\sigma \in \widehat{G} : \sigma|_H = 1\}$ is the *annihilator* of H. Deduce that H^A is compact and open.

   (b) Verify that $(p^n 1_{p^n\mathbb{O}_p})_{n=1}^{\infty}$ is a special summability kernel for $\mathbb{Q}_p$, and also for $\mathbb{O}_p$.

3. (a) Let $\{C_k\}_{k=1}^{\infty}$ be a sequence of finite cyclic groups and $P = \prod_{k=1}^{\infty} C_k$ be its compact product, and $S = \bigoplus_{k=1}^{\infty} C_k$ be its discrete direct sum. Verify the relations

   $$\widehat{P} \cong S, \quad \widehat{S} \cong P \text{ (isomorphically \& homeomorphically).}$$

   (b) Exhibit a special summability kernel for P.

   (c) Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (p prime). Compute the dual of the additive group of the topological field $\mathbb{F}_p(\!(X)\!)$. [Hint: $\mathbb{F}_p(\!(X)\!) = S \oplus \mathbb{F}_p[[X]]$, where S represents the "principal parts".]

4. Let $(e_n)_{n=1}^{\infty}$ be a summability kernel for G.

   (a) Let $f \in L^1 \cap \mathcal{C}_0(G)$. Show that $\lim_{n\to\infty}\|e_n * f - f\|_\infty = 0$. [Hint: consider f in $\mathcal{C}_c(G)$, first.]

   (b) Given a unitary representation $\pi : G \to \mathrm{U}(\mathcal{H}_\pi)$, show that w.o.t.-$\lim_{n\to\infty}\pi(x * e_n) = \pi(x)$.

5. For this question, we let G and G' be countable discrete groups. Given separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we define an inner product on the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ by

$$\left\langle \sum_{i=1}^{n} v_i \otimes w_i, \sum_{j=1}^{n'} v'_j \otimes w'_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n'} \langle v_i, v'_j \rangle_{\mathcal{H}} \langle w_i, w'_j \rangle_{\mathcal{K}}$$

and let $\mathcal{H} \otimes^2 \mathcal{K}$ be the Hilbert space which is the completion. Notice that if $(e_i)_{i=1,2,\ldots}$ and $(f_j)_{j=1,2,\ldots}$, are orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively, then $(e_i \otimes f_j)_{i=1,2,\ldots,j=1,2\ldots}$ is orthonormal and dense in $\mathcal{H} \otimes \mathcal{K}$. It follows that the Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathcal{H} \otimes \mathcal{K}$ is non-degenerate, and thus an inner product. [Verify this for yourself, but do not submit it.] Furthermore, $(e_i \otimes f_j)_{i=1,2,\ldots,j=1,2\ldots}$ is an orthonormal basis for $\mathcal{H} \otimes^2 \mathcal{K}$.

If $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{K})$ we let $S \otimes T \in \mathcal{B}(\mathcal{H} \otimes^2 \mathcal{K})$ be the unique linear operator which satisfies

$$S \otimes T(v \otimes w) = Sv \otimes Tw \text{ for } v \in \mathcal{H} \text{ and } w \in \mathcal{K}.$$

If θ, θ' are unitary representation of G, G', respectively, define the *Kronecker product representation*

$$\theta \times \theta' : G \times G' \to \mathrm{U}(\mathcal{H}_\theta \otimes^2 \mathcal{H}_{\theta'}),\ \theta \times \theta'(s, s') = \theta(s) \otimes \theta'(s')$$

and, if $G = G'$, define the *tensor product representation*

$$\theta \otimes \theta' : G \to \mathrm{U}(\mathcal{H}_\theta \otimes^2 \mathcal{H}_{\theta'}),\ \theta \otimes \theta'(s) = \theta(s) \otimes \theta'(s).$$

Hence $\theta \otimes \theta' = \theta \times \theta'|_{G_D}$ where $G_D = \{(s, s); s \in G\}$.

(a) Show: $\ell^2(G) \otimes^2 \mathcal{H} \cong \ell^2(G, \mathcal{H})$, $\ell^2(G) \otimes^2 \ell^2(G') \cong \ell^2(G \times G')$.

(b) Let K be a finite subgroup of G and π be a unitary representation of K (on a separable Hilbert space). Define

$$P_\pi \text{ in } \mathcal{B}(\ell^2(G) \otimes^2 \mathcal{H}_\pi) \text{ by } P_\pi = \frac{1}{|K|} \sum_{k \in K} \rho(k) \otimes \pi(k)$$

where $\rho : G \to \mathrm{U}(\ell^2(G))$ is the right regular representation. Show that $P_\pi^2 = P_\pi^* = P_\pi$ and that

$$\ell^2(G) \otimes_K^2 \mathcal{H}_\pi = P_\pi(\ell^2(G) \otimes^2 \mathcal{H}_\pi)$$

is $\lambda(s) \otimes I$-invariant for each s in G. Let

$$\Lambda^\pi = \lambda(\cdot) \otimes I|_{\ell^2(G)\otimes^2_K \mathcal{H}_\pi} : G \to \mathrm{U}(\ell^2(G) \otimes^2_K \mathcal{H}_\pi).$$

Show that $\Lambda^\pi \cong \mathrm{ind}_H^G \pi$ (unitary equivalence).
[Notation $\delta_s \otimes_K v := P_\pi(\delta_s \otimes v)$ may come in handy: e.g. $\delta_{sk} \otimes_K v = \delta_s \otimes_K \pi(k)v$.]

(c) (Induction in stages.) Let $K \subset H$ be finite subgroups of G and π a unitary representation of K. Show that $\mathrm{ind}_H^G \mathrm{ind}_K^H \pi \cong \mathrm{ind}_K^G \pi$. [First, show that $\ell^2(G) \otimes^2_H (\ell^2(H) \otimes^2_K \mathcal{H}_\pi) \cong \ell^2(G) \otimes^2_K \mathcal{H}_\pi$.]

(d) (Induction of Kronecker products.) Let $K \subset G$, $K' \subset G'$ be finite subgroups, and π, π' be unitary representations of K, K' respectively. Show that $\mathrm{ind}_{K\times K'}^{G\times G'}(\pi \times \pi') \cong \mathrm{ind}_K^G \pi \times \mathrm{ind}_{K'}^{G'} \pi'$.

(e) (Induction of tensor products.) Let K be a finite subgroup of G and π, π' be unitary representations of K. Show that $\mathrm{ind}_K^G(\pi \otimes \pi') \tilde{\subset} \mathrm{ind}_K^G \pi \otimes \mathrm{ind}_K^G \pi'$, i.e. the intertwiner space $\mathcal{I}_G(\mathrm{ind}_K^G(\pi \otimes \pi'), \mathrm{ind}_K^G \pi \otimes \mathrm{ind}_K^G \pi')$ contains an isometry.

6. In each of the cases below, H is an open abelian subgroup of G. For each example below, consider the family of representations $\{\mathrm{ind}_H^G \chi : \chi \in \widehat{H}\}$. Classify these into unitary equivalence classes and indicate which are irreducible.

(a) $H = \mathbb{Z}$, $G = \mathbb{Z} \rtimes \{-1, 1\}$, with action $\alpha(m) = \alpha m$ for $\alpha \in \{-1, 1\}$, $m \in \mathbb{Z}$.

(b) $H = \mathbb{Z}^2$, $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$, with action $m \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} k + ml \\ l \end{bmatrix}$, for k, l, m in $\mathbb{Z}$.

(c) $H = \mathbb{T}^2$, $G = \mathbb{T}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, with action $\alpha \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} w^a z^b \\ w^c z^d \end{bmatrix}$, for $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ and w, z in $\mathbb{T}$. [The orbit of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\mathbb{Z}^2$ consists of all relatively prime pairs.]

(d) $H = (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}}$ (countable direct sum group), $G = H \rtimes \mathbb{Z}$, with action $m(x_k)_{k\in\mathbb{Z}} = (x_{k+m})_{k\in\mathbb{Z}}$ (i.e. shift in coordinates) for $(x_k)_{k\in\mathbb{Z}}$ in H and m in $\mathbb{Z}$.