PMATH 950, Fall 2018

Assignment #1 Due: October 9.

1. Let $\mathbb{K}$ be a non-discrete locally compact field whose topology is induced by the metric associated with a field norm $|\cdot| : \mathbb{K} \to [0, \infty)$ (e.g. usual absolute value for $\mathbb{K} = \mathbb{R}$, modulus for $\mathbb{K} = \mathbb{C}$, or p-adic norm in the case $\mathbb{K} = \mathbb{Q}_p$).

   Given n in $\mathbb{N}$, fix a $\mathbb{K}$-*vector space norm* $\|\cdot\| : \mathbb{K}^n \to [0, \infty)$, i.e. so for $x, y \in \mathbb{K}^n$ and α in $\mathbb{K}$ we have

   $$\|x\| = 0 \iff x = 0, \quad \|x + y\| \leq \|x\| + \|y\|, \quad \text{and } \|\alpha x\| = |\alpha|\|x\|$$

   and the further normalization property that

   $$\{\|x\| : x \in \mathbb{K}^n\} = \{|a| : a \in \mathbb{K}\}$$

   (e.g. Euclidean norm in the case $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or norm $\|x\| = \max_{j=1,\ldots,n} |x_j|_p$ in the case $\mathbb{K} = \mathbb{Q}_p$).

   (a) Show that $\bar{B}(\mathbb{K}^n) = \{x \in \mathbb{K}^n : \|x\| \leq 1\}$ is compact.

   (b) Show that the map $|\!|\!|\cdot|\!|\!| : \mathrm{M}_n(\mathbb{K}) \to [0, \infty)$ given by

   $$|\!|\!|A|\!|\!| = \max_{x \in \bar{B}(\mathbb{K}^n)} \|Ax\|$$

   is a $\mathbb{K}$-vector space norm with the further property that for A, B in $\mathrm{M}_n(\mathbb{K})$ we have that

   $$|\!|\!|AB|\!|\!| \leq |\!|\!|A|\!|\!|\,|\!|\!|B|\!|\!|.$$

   We say that $|\!|\!|\cdot|\!|\!|$ is a $\mathbb{K}$-*algebra norm* on $\mathrm{M}_n(\mathbb{K})$.

   (c) Show that the function $\rho : \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}) \to [0, \infty)$ given by

   $$\rho(a, b) = \log(1 + |\!|\!|a^{-1}b - I|\!|\!| + |\!|\!|b^{-1}a - I|\!|\!|)$$

   defines a left-invariant metric on $\mathrm{GL}_n(\mathbb{K})$.

   (d) Show that the metric topology τ_ρ coincides with the relativized product topology $\pi = \tau_{\mathbb{K}}^{\times n^2}|_{\mathrm{GL}_n(\mathbb{K})}$, where $\tau_{\mathbb{K}}$ is the topology on $\mathbb{K}$ and we identify $\mathrm{M}_n(\mathbb{K}) \cong \mathbb{K}^{n^2}$.

2. Let G be a locally compact group whose topology is given by a right-invariant metric, i.e. a metric $\rho : G \times G \to [0, \infty)$ for which $\rho(xz, yz) = \rho(x, y)$ for x, y, z in G.

   (a) Show that any ρ-Cauchy sequence in G admits a limit point.

   (b) Show that given a closed subgroup H, the map $d : G/H \times G/H \to [0, \infty)$ given by
   $$d(xH, yH) = \inf\{\rho(x, yh) : h \in H\}$$
   is a metric, which gives the quotient topology on G/H and for which $(G/H, d)$ is complete.

   (c) Show that $\check{\rho} : G \times G \to [0, \infty)$, given by $\check{\rho}(x, y) = \rho(x^{-1}, y^{-1})$ is left-invariant, and induces the topology on G.

   (d) The metrics ρ and ρ' are *equivalent* if there are $m, M > 0$ such that $m\rho \leq \rho' \leq M\rho$.

   Show that if the right invariant metric ρ on G is equivalent to a left invariant metric ρ', then there is a neighbourhood V of e for which $\overline{\bigcup_{x \in G} xVx^{-1}}$ is compact.

   (e) Deduce that $\mathrm{GL}_2(\mathbb{R})$ admits no pair of equivalent metrics, one right-invariant and one left-invariant, which induce the topology.

3. Let G be a locally compact group.

   (a) Let $V = V^{-1}$ be an open neighbourhood of e in G. Verify that $H = \bigcup_{n=1}^{\infty} V^n$ is an open subgroup of G.

   (b) Determine all of the implications between the conditions below. If one condition does no imply another, demonstrate via a counterexample.

   • G is *σ-compact*: there is a sequence of compact subsets $(K_n)_{n=1}^{\infty}$ of G for which $G = \bigcup_{n=1}^{\infty} K_n$;

   • G is *compactly generated*: there is a compact set K for which the smallest closed subgroup $\overline{\langle K \rangle}$ of G, containing K, is G; and

   • $(G, \mathcal{B}(G), m)$ (left Haar measure) is *σ-finite*: there is a sequence $(E_n)_{n=1}^{\infty} \subseteq \mathcal{B}(G)$ such that each $m(E_n) < \infty$ and $G = \bigcup_{n=1}^{\infty} E_n$.

   • G is *connected*: if $G = U \cup V$ where each of U and V are open and non-empty, then $V \cap U \neq \varnothing$.

(c) Show that if G metrizable and σ-compact, then it is second countable.

(d) Let G be σ-compact, and let (X, d) be a complete metric space which is a *transitive* G-space, i.e. for some x in X, $G \cdot x = \{s \cdot x : s \in G\}$ is all of X. Let $S_x = \{s \in G : s \cdot x = x\}$. Show that the map $\eta : G/S_x \to X$, $\eta(sS_x) = s \cdot x$ is a homeomorphism.