PMATH 950, Winter 2016

Assignment #3 Due: March 24.

Generally, G will denote a locally compact group, below.

1. Let G be compact and $s \in G$. Show that $\overline{\{s^n : n \in \mathbb{N}\}}$ is a subgroup of G. Deduce that any closed subsemigroup of G is necessarily a subgroup.

2. Let $G = \mathbb{T} \rtimes \{1, -1\}$ with group law
$$(z, a)(w, b) = (zw^a, ab).$$
Verify that the list of representations
$$1 \text{ (trivial)}, \ (z, a) \mapsto a, \ \text{ and } \pi_n(z, a) = \begin{bmatrix} z^n & 0 \\ 0 & \bar{z}^n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{(1-a)/2} \quad (n \in \mathbb{N})$$
is a list of irreducible representations, and is all of $\widehat{G}$.

3. Let G be compact. A *sub-hypergroup* of $\widehat{G}$ is a subset S for which:
(a) $\bar{\pi} \in S$ if $\pi \in S$; and
(b) for $\pi, \pi' \in S$, $\pi \otimes \pi' = \pi_1 \oplus \cdots \oplus \pi_n$ for $\pi_1, \ldots, \pi_n$ in S.
Show that the sub-hypergroups S of $\widehat{G}$ are the sets of the form
$$S_N = \{\pi \in \widehat{G} : N \in \ker \pi\} \cong \widehat{G/N}$$
for a closed normal subgroup N of G.

4. Let $\{G_i\}_{i \in I}$ be a non-empty family of compact groups and $G = \prod_{i \in I} G_i$ its product. Show that $\widehat{G}$ is of the form
$$(x_i)_{i \in I} \mapsto \pi_{i_1}(x_{i_1}) \otimes \cdots \otimes \pi_{i_n}(x_{i_n}) : G \to \mathcal{U}(\mathcal{H}_{\pi_{i_1}} \otimes \cdots \otimes \mathcal{H}_{\pi_{i_n}})$$
where $\pi_{i_j} \in \widehat{G_{i_j}}$ for some distinct $i_1, \ldots, i_n$ in I, $n \in \mathbb{N}$. Deduce that if each G_i is abelian, then $\widehat{G} = \sum_{i \in I} \widehat{G_i}$ (algebraic direct sum of abelian groups). [You may wish to use the fact that $m_G = \bigtimes_{i \in I} m_{G_i}$ (Radon product of probability measures on $\mathcal{B}(G)$).]

5. Let G be compact.

   (a) Show that $\gamma : G \to \prod_{\pi \in \widehat{G}} \mathcal{U}(\mathcal{H}_\pi)$, $\gamma(x) = (\pi(x))_{\pi \in \widehat{G}}$ is a homeomorphism onto its range.

   (b) Deduce that G is metrizable if and only if $\widehat{G}$ is countable.

   (c) We say that G admits *small subgroups* if for any neighbourhood U of e, there is a subgroup $H \subset U$. Show that if G admits no small subgroups, then it is isomorphic to closed subgroup of a unitary group $\mathcal{U}(\mathbb{C}^n)$ for some n.

   Note: the converse to (c) is also true, but its proof requires some Lie theory.

6. **(a)** Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a bounded finite dimensional representation of a non-compact group. Verify that $\overline{\pi(G)}$ is a compact subgroup of $\mathcal{B}(\mathcal{H})$ and hence similar to a subgroup of $\mathcal{U}(\mathcal{H})$.

   (b) Let

   $\mathrm{Irr}_{\mathrm{fin}}(G) = \{\pi : G \to \mathcal{U}(\mathbb{C}^d)$ bounded continuous representation, $d \in \mathbb{N}\}$

   and $\widehat{G}_{\mathrm{fin}} = \mathrm{Irr}_{\mathrm{fin}}(G)/\sim$, where $\sim$ is relation of unitary equivalence. Then let

   $$G^{ap} = \overline{\{(\pi(x))_{\pi \in \widehat{G}_{\mathrm{fin}}} : x \in G\}} \subseteq \prod_{\pi \in \widehat{G}_{\mathrm{fin}}} \mathcal{U}(\mathbb{C}^{d_\pi})$$

   denote the *almost periodic compactification* of G. Verify that this is the universal compact group containing a dense continuous image of G:

   > if $\eta : G \to K$ is a continuous homomorphism into a compact group, then there is a unique continuous map $\eta^{ap} : G^{ap} \to K$ such that $\eta^{ap}(\gamma(x)) = \eta(x)$ for x in G, where $\gamma(x) = (\pi(x))_{\pi \in \widehat{G}_{\mathrm{fin}}}$.

   (c) Let $\mathcal{AP}(G) = \{f \circ \gamma : f \in \mathcal{C}(G^{ap})\}$. Show that for f in $\mathcal{C}_b(G)$ $f \in \mathcal{AP}(G)$ if and only if the norm closure of the orbit $G * f = \{x * f : x \in G\}$ is compact in $\mathcal{C}_b(G)$. [If $\overline{G * f}$ is compact, consider the strong operator closure of the group generated by left translation operators $\{L(x) : x \in G\}$ on $\overline{\mathrm{span}} G * f \subseteq \mathcal{C}_b(G)$.]

   (d) Show that G is compact if and only if $\mathcal{AP}(G) = \mathcal{C}_b(G)$. [Hint: consider $\mathcal{C}_c(G)$ when G is not compact.]