PMATH 950, Winter 2016

Assignment #2 Due: February 25.

Generally, G will denote a locally compact group, below.

1. Let N be a closed normal subgroup of G.

   (a) Verify that the funtional on $\mathcal{C}_c(G)$ given by
$$I(f) = \int_{G/N} \int_N f(xn)\, dn\, dxN$$
   where $dn = dm_N(n)$, $dxN = dm_{G/N}(xN)$ for choices of left Haar measures m_N and $m_{G/N}$, defines a left Haar integral.

   [Hint: Let $T_N : \mathcal{C}_c(G) \to \mathcal{C}_c(G/N)$ be given by $T_N f(xN) = \int_N f(xn)\, dn$. The above formula takes the form $\int_{G/N} T_N f(xN)\, dxN$.]

   (b) Let $\Delta_N : N \to (0, \infty$, and $\Delta_G : G \to (0, \infty)$ denote the respective modular functions. Deduce that $\Delta_G|_N = \Delta_N : N \to (0, \infty)$.

   (c) Let $Z(G) = \{z \in G : xz = zx \text{ for } x \text{ in } G\}$. [We may take for granted the easy fact that this is a closed subgroup.] Show that if $G/Z(G)$ is unimodular, then soo too is G.

   (d) Suppose G admits a compact normal subgroup K for which G/K is unimodular. Then G is unimodular as well. [You may wish to verify that if α is a continuous automorphism of K, then $\delta_K(\alpha) = 1$, in notation of A1, Q4.]

   (e) Verify that the following two groups are unimodular, and compute Haar measures for each:
$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}, \quad G = \mathbb{T}^d \rtimes \mathrm{SL}_d(\mathbb{Z})$$
   where the product in G is given by
$$(z, a)(w, b) = (z\, a{\cdot}w, ab), \text{ where } [a_{ij}]{\cdot}(z_1, \dots, z_d) = \left(\prod_{j=1}^d z_j^{a_{1j}}, \dots, \prod_{j=1}^d z_j^{a_{dj}} \right)$$
   [Recognizing that $\mathbb{T}^d \cong \mathbb{R}^d/(2\pi\mathbb{Z})^d$ helps to see that the latter is a group.]

2. **(a)** Define for E in $\mathcal{B}(G)$

$$\widetilde{E} = \{(x,y) \in G \times G : xy \in E\}.$$

Show that $\widetilde{E} \in \mathcal{B}(G \times G)$. Define for μ, ν in $M(G)$

$$\mu \tilde{*} \nu(E) = \mu \times \nu(E).$$

Show that $\mu \tilde{*} \nu$ is a Radon measure such that for φ in $\mathcal{C}_c(G)$ that

$$\int_G \varphi \, d(\mu \tilde{*} \nu) = \int_G \varphi \, d(\mu * \nu)$$

and hence $\mu \tilde{*} \nu = \mu * \nu$. [Using Jordan decomposition, you may elect to work with positive measures. You may use, without proof, the fact that the product of finite Radon measures is Radon.]

(b) Deduce, for E, μ and ν as above that

$$\mu * \nu(E) = \int_G \nu(x^{-1}E) \, d\mu(x) = \int_G \mu(Ey^{-1}) \, d\nu(y).$$

Explain why $x \mapsto \nu(x^{-1}E) : G \to \mathbb{C}$ is Borel measurable.

(c) Let $M_+^1(G) = \{\mu \in M(G) : \mu \geq 0 \text{ and } \mu(G) = 1\}$ denote the space of probability measures. Show that for μ, ν in $M_+^1(G)$ that $\mu * \nu \in M_+^1(G)$ too.

3. Let N be a closed normal subgroup in G. Let

$$\tau_{G_N} = \{U \subseteq G : x^{-1}(U \cap xN) \in \tau_N \text{ for all } x \in G\}$$

where τ_N is the relitivized topology from G on N.

Example: Notice that $\tau_{G_{\{e\}}}$ is the discrete topology on G. If N is open, then $\tau_{G_N} = \tau_G$.

(a) Show that $G_N = (G, \tau_{G_N})$ is a locally compact group, whose topology is finer than τ_G.

(b) Show that if H is a locally compact group, and $\eta : H \to G$ is a continuous homomorphism, then the map $M(\eta) : M(H) \to M(G)$ given by

$$\int_G \varphi \, d[M(\eta)(\mu)] = \int_H \varphi \circ \eta \, d\mu(h), \text{ where } \mu \in M(H), \ \varphi \in \mathcal{C}_c(G)$$

defines a contractive homomorphism, which is isometric if η is injective.

(c) If $\iota : G_N \to G$ and $\jmath : N \to G$ be the formal identity maps, show that

$$M(\iota)[M(G_N)] = \ell^1\text{-}\bigoplus_{x\in T} \delta_x * M(\jmath)[M(N)]$$

where T is a set of distinct coset representatives of N in G.

(d) Let

$$I(G_N) = \{\mu \in\ M(G) : \mu(xE) = 0 \text{ whenever } E \in \mathcal{B}(N),\ x \in G\}.$$

Show that $I(G_N)$ is an ideal in $M(G)$.

4. Let G be abelian, H be a closed subgroup of G. We let $H^a = \{\sigma \in \widehat{G} : \sigma(s) = 1 \text{ for all } x \text{ in } H\}$ denote the *annihilator* of H in $\widehat{G}$.

   (a) Verify that H^a is a closed subgroup of $\widehat{G}$.

   (b) Show that there is a natural isomorphism: $\widehat{H} \cong \widehat{G}/H^a$.

   (c) Show that there is a natural isomorphism: $\widehat{G/H} \cong H^a$.

   (d) Deduce that H is open in G if and only if H^a is compact; and H is compact if and only if H^a is open in $\widehat{G}$.

5. Let p be a prime.

   (a) Let $\mathbb{T}_p = \{z \in \mathbb{T} : z^{p^n} = 1 \text{ for some } n \text{ in } \mathbb{N}\}$ be the p-power torsion subgroup of $\mathbb{T}$. Show that there is a natural isomorphism $\widehat{\mathbb{O}_p} \cong \mathbb{T}_p$. [Hint: any character on $\mathbb{O}_p$ is determined by its values on the dense copy of $\mathbb{Z}$.]

   (b) Show that there is a natural homeomorphic isomorphism $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$. [Consider the basic character σ, given by $\sigma_1|_{\mathbb{O}_p} = 1$, and for $x \in \mathbb{Q}_p \setminus \mathbb{O}_p$

$$\sigma_1(x) = e^{2\pi i \sum_{j=-m}^{-1} a_j p^j}, \text{ where } x = \sum_{j=-m}^{\infty} a_j p^j,\ a_j \in \{0, 1, \ldots, p-1\}.$$

   If $\xi \in \mathbb{Q}_p$, let $\sigma_\xi(x) = \sigma_1(\xi x)$, which is a character. Conversely, any character σ_1 is determined by what it does on each subgroup $\mathbb{O}_p \subset \frac{1}{p}\mathbb{O}_p \subset \frac{1}{p^2}\mathbb{O}_p \subset \ldots$.]

   (c) Show that $\mathbb{O}_p^a \cong \mathbb{O}_p$ in $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$.