PMATH 950, Winter 2016

Assignment #1 Due: January 28.

Unless otherwise stated, (G,τ) always denotes a Hausdorff locally compact group.

1. Show that (G,τ) is *complete* in the following sense: If (x_α) is a net in G which satisfies the property that for every V in τ such that $e \in V$, the is α_V such that $x_\alpha^{-1}x_\beta \in V$ for $\alpha,\beta \geq \alpha_V$, then there is x_0 in G such that $\lim_\alpha x_\alpha = x_0$.

   [An analogous statement holds with $x_\alpha x_\beta^{-1} \in V$ for $\alpha,\beta \geq \alpha_V$, as well.]

2. **(a)** Let $U \in \tau$ satisfy that $\overline{U}$ is compact. Prove that $\overline{U}$ is either finite or uncountable.

   (b) Deduce that the only Hausdorff topology σ on a countable group Γ which allows (Γ,σ) to be a locally compact group is the discrete topology.

   (c) Exhibit an example of a countable topological group which is not locally compact.

3. **(a)** Let $U \in \tau$ with $e \in U$. Show that $H = \bigcup_{n=1}^{\infty} U^n$ contains an open subgroup of G. Deduce that if G is connected, it is compactly generated, i.e. there is a compact set L for which the smallest subgroup containing L is all of G.

   (b) Suppose there is $U \in \tau$ with $e \in U$ and U itself is compact. Prove that U contains a compact open subgroup K of G. [Hint: show that continuity of multiplication allows us to find neighbourhood V of e for which $VU \subseteq U$.]

   (c) Show that if (G,τ) is totally disconnected, then every U in τ with $e \in U$ contains a compact V in τ with $e \in V$. Deduce that a there is a basis for τ at e consisting of compact open subgroups if and only if (G,τ) is totally disconnected.

   (d) Show that if (G,τ) is totally disconnected and compact, then there is a basis $\mathcal{N}$ for τ at e consisting of open normal subgroups. Deduce that G embeds in a product of finite groups, and that τ is metrizable only if $\mathcal{N}$ can be arranged to be countable. [Hint: show that if K

is an open subgroup, then $\bigcap_{x\in G} xKx^{-1}$ may be realised as a finite intersection of conjugates of K.]

(e) Show that if (G,τ) is totally disconnected anf H is a closed normal subgroup of G, then $(G/H, \tau_{G/H})$ (quotient topology) is totally disconnected.

4. Let (A,σ) be a locally compact group. We say that (A,σ) acts continuously on (G,τ) if for α in A, $x \mapsto \alpha(x)$ is an automorphism and the map $(x,\alpha)\mapsto\alpha(x) : G\times A \to G$ is $\tau\times\sigma-\tau$ continuous. Let m_G denote the left Haar measure on G.

   (a) Show that there is a continuous homomorphism $\delta : A \to (0,\infty)$ defined by $\delta(\alpha)m_G(E) = m_G(\alpha(E))$ for $E \in \mathcal{B}(G)$.

   (b) Define the semi-direct product of G by A by

   $$G \rtimes A = G\times A \text{ (as a set), with product } (x,\alpha)(y,\beta) = (x\alpha(y), \alpha\beta).$$

   Verify that this is a locally compact group and that

   $$\int_{G\rtimes A} f\,dm = \int_G\int_A f(x,\alpha)\,\frac{dm_A(\alpha)}{\delta(\alpha)}\,dm_G(x),\ \ f\in\mathcal{C}_c(G\rtimes A)$$

   defines a left Haar integral.

   (c) Compute formulas for both left and right Haar integrals on

   $$H = \left\{\begin{bmatrix} a & x\\ 0 & 1\end{bmatrix} : a\in \mathrm{GL}_n(\mathbb{R}), x\in\mathbb{R}^n \text{ (column vectors)}\right\} \subset \mathrm{GL}_{n+1}(\mathbb{R}).$$

5. A *summability kernel* is a net (k_α) in $L^1(G)$ which satisfies

   - $\lim\limits_\alpha \int_G k_\alpha\,dm = 1$
   - $\sup_\alpha \|k_\alpha\|_1 < \infty$, and
   - for each V in τ with $e\in V$, $\lim\limits_\alpha \int_{G\setminus V} |k_\alpha|\,dm = 0.$

   (a) Show that $\lim_\alpha \|k_\alpha * f - f\|_1 = 0$ for each f in $L^1(G)$.

   (b) Let $\mathcal{V}$ be a basis for τ at e, and let $V\le V'$ in $\mathcal{V}$ if and only if $V\supseteq V'$. Show that $(\frac{1}{m(V)}1_V)_{V\in\mathcal{V}}$ is a summability kernel.