PMATH 950 (833), Winter 2018

Assignment #3 Due: April 3.

1. Let G be abelian, H be a closed subgroup of G. We let $H^a = \{\sigma \in \widehat{G} : \sigma(s) = 1 \text{ for all } x \text{ in } H\}$ denote the *annihilator* of H in $\widehat{G}$.

   (a) Verify that H^a is a closed subgroup of $\widehat{G}$.

   (b) Show that there is a natural isomorphism: $\widehat{G/H} \cong H^a$.

   (c) Show that there is a natural isomorphism: $\widehat{H} \cong \widehat{G}/H^a$.

   (d) Deduce that H is open in G if and only if H^a is compact; and H is compact if and only if H^a is open in $\widehat{G}$.

   [Notice that (b) and (c) tell us that $\mathbb{Z}/n\mathbb{Z}$ and $C_n = \langle e^{2\pi i/n} \rangle \subset \mathbb{T}$ are a dual pair. This is, of course, simple to prove manually. Inversion Theorem, on this pair, is frequently used in analytic number theory.]

2. Let $\alpha : G \to H$ be a continuous homomorphism between locally compact abelian groups, and $\alpha^* : \widehat{H} \to \widehat{G}$ be given by $\alpha^*(\sigma) = \sigma \circ \alpha$.

   (a) Show that $\alpha^* : \widehat{H} \to \widehat{G}$ is continuous.

   (b) Find conditions on α^* which characterize when α is injective, or when α admits dense range.

3. Let p be a prime.

   (a) Let $\mathbb{T}_p = \{z \in \mathbb{T} : z^{p^n} = 1 \text{ for some } n \text{ in } \mathbb{N}\}$ be the p-power torsion subgroup of $\mathbb{T}$. Show that there is a natural isomorphism $\widehat{\mathbb{O}_p} \cong \mathbb{T}_p$. [Hint: any character on $\mathbb{O}_p$ is determined by its values on the dense copy of $\mathbb{Z}$.]

   (b) Show that there is a natural homeomorphic isomorphism $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$. [Show the basic character σ, given by $\sigma_1|_{\mathbb{O}_p} = 1$, and for $x \in \mathbb{Q}_p \backslash \mathbb{O}_p$

   $$\sigma_1(x) = e^{2\pi i \sum_{j=-m}^{-1} a_j p^j}, \text{ where } x = \sum_{j=-m}^{\infty} a_j p^j,\ a_j \in \{0, 1, \dots, p-1\}$$

   is continuous. If $\xi \in \mathbb{Q}_p$, let $\sigma_\xi(x) = \sigma_1(\xi x)$. Conversely, any character σ_1 is determined by what it does on each subgroup $\frac{1}{p^k}\mathbb{O}_p$.]

   (c) Deduce that $\mathbb{O}_p^a \cong \mathbb{O}_p$ in $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$.

4. Let G be a locally compact abelian group and $U : L^2(G) \to L^2(\widehat{G})$ be the Plancherel unitary. We also let $\lambda : G \to \mathrm{U}(L^2(G))$ denote the *left regular representation*

$$\lambda(x)f = x * f, \quad x * f(y) = f(x^{-1}y) \text{ for a.e. } y \text{ in } G.$$

   (a) Let $\hat{\lambda} : G \to \mathrm{U}(L^2(\widehat{G}))$ be given by

   $$\hat{\lambda}(x)f(\sigma) = \bar{\sigma}(x)f(\sigma) \text{ for a.e. } \sigma \text{ in } \widehat{G}.$$

   Show that $U\lambda(x)U^* = \hat{\lambda}(x)$ for x in G, i.e. λ and $\widehat{\lambda}$ are *unitarily equivalent*.

   (b) Show that the weak operator closure of $\mathrm{span}\hat{\lambda}(G)$ contains the multiplication operators M_φ for $\varphi \in L^\infty(\widehat{G})$. [Use a Hahn-Banach separation argument; implemented using an idea is very similar to one of in proof that $\widehat{G}$ is a topological group, given in lectures.]

   (c) Deduce that if G is compact $\Leftrightarrow$ $L^2(G)$ admits a finite dimensional λ-invariant subspace. [If $\mathcal{L}$ is a $\hat{\lambda}$-invariant subspace, show that there is a smallest closed subset F of $\widehat{G}$ for which $M_{1_F}P_{\mathcal{L}} = P_{\mathcal{L}}$.]

5. Let G be compact. A *sub-hypergroup* of $\widehat{G}$ is a subset S for which:

   (i) $\bar{\pi} \in S$ if $\pi \in S$; and

   (ii) for $\pi, \pi' \in S$, $\pi \otimes \pi' = \pi_1 \oplus \cdots \oplus \pi_n$ for some $\pi_1, \ldots, \pi_n$ in S.

   Show that the sub-hypergroups S of $\widehat{G}$ are the sets of the form

   $$S_N = \{\pi \in \widehat{G} : N \subseteq \ker \pi\} \cong \widehat{G/N}$$

   indexed over closed normal subgroups N of G.

6. Let $\{G_i\}_{i \in I}$ be a non-empty family of compact groups and $G = \prod_{i \in I} G_i$ its product.

   (a) Show that $\widehat{G}$ is of the form

   $$(x_i)_{i \in I} \mapsto \pi_{i_1}(x_{i_1}) \otimes \cdots \otimes \pi_{i_n}(x_{i_n}) : G \to \mathrm{U}(\mathcal{H}_{\pi_{i_1}} \otimes \cdots \otimes \mathcal{H}_{\pi_{i_n}})$$

   where $\pi_{i_j} \in \widehat{G_{i_j}}$ for some distinct $i_1, \ldots, i_n$ in I, $n \in \mathbb{N}$.

   (b) Deduce that if each G_i is abelian, then $\widehat{G} \cong \bigoplus_{i \in I} \widehat{G_i}$ (algebraic direct sum of abelian groups).

   [There is an evident related fact, for locally compact abelian groups, that $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$.]

7. Let now $\{G_i\}_{i\in I}$ be a family of locally compact abelian groups, each admitting a given compact open subgroup K_i. Define the *restricted direct product* over $\{K_i\}_{i\in I}$ by

$$H = \prod_{i\in I}(G_i, K_i) = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} G_i : \text{ all but finitely many } x_i \in K_i \right\}.$$

Let $K = \prod_{i\in I} K_i$, and equip H with the topology having base $\{xU : x \in H, U \in \tau_K\}$.

(a) Show that this group is locally compact and that there is a natural homeomorphic isomorphism

$$\widehat{H} \cong \prod_{i\in I}(\widehat{G_i}, K_i^a).$$

[Notice that $H = \bigcup_{F\subset G \text{ finite}} H_F$, each $H_F = \prod_{i\in F} G_i \times \prod_{i\in I\setminus F} K_i$.]

(b) Show that $\mathbb{A} = \prod_{p \text{ prime}}(\mathbb{Q}_p, \mathbb{O}_p)$ is a topological ring. Compute the unit group $\mathbb{A}^\times$.

(c) Show that the diagonal embedding $n \mapsto (n\mathbf{1}_2, n\mathbf{1}_3, n\mathbf{1}_5, \dots) : \mathbb{Z} \to \prod_{p \text{ prime}} \mathbb{O}_p$ has dense range, and extends to a homomorphism of $\mathbb{Q}$ into $\mathbb{A}$ with dense range.

8. (Convergence of convolution powers to Haar measure, revisited.) Let G be a compact group and μ be a probability measure on G with $\langle \mathrm{supp}\mu \rangle$ dense in G. Show that $\mu_n = \frac{1}{n}\sum_{k=1}^n \mu^{*k}$ converges to the normalized Haar measure m.

[This time, let $\pi \in \widehat{G}\setminus\{1\}$. Show that $\widehat{\mu_n}(\pi)$ converges to 0, unless $\hat{\mu}(\pi)$ admits 1 as an eigenvalue. In that case, show that for any associated eigenvector ξ, $\langle \xi | \pi(\cdot)\xi \rangle$ must be constant on G, and this contradicts irreducibility of π if $d_\pi > 1$. Using representation theory is more fun.]