PMATH 950, Winter 2016

Assignment #2 Due: February 27.

Generally, G will denote a locally compact group, below.

1. A net $(f_\alpha)_\alpha \subset L^1(G)$ is called a *summability kernel* provided that
   - $\sup\limits_\alpha \|f_\alpha\|_1 < \infty$ (we say that $(f_\alpha)_\alpha$ is *contractive* if each $\|f_\alpha\|_1 \leqslant 1$);
   - $\lim\limits_\alpha \int_G f_\alpha \, dm = 1$; and
   - for each open neighbourhood U of e, $\lim\limits_\alpha \int_{G \setminus U} |f_\alpha| \, dm = 0$.

   (a) Let $(U_\alpha)_\alpha$ be a decreasing net of open sets (i.e. $\alpha \leqslant \alpha'$ implies $U_\alpha \supseteq U_{\alpha'}$) which base for the topology at e. Show that $(\frac{1}{m(U_\alpha)} 1_{U_\alpha})_\alpha$ is a contractive summability kernel.

   (b) Let $\mathcal{X}$ be a Banach G-module. Show that for ξ in $\mathcal{X}$

   $$f_\alpha \cdot \xi = \int_G f_\alpha(x) \, x \cdot \xi \, dx \text{ (Bochner integral)}$$

   satisfies $\lim_\alpha f_\alpha \cdot \xi = \xi$. Deduce that (f_α) is a *bounded approximate identity* for $L^1(G)$: $\lim_\alpha f_\alpha * f = f = \lim_\alpha f * f_\alpha$, for f in $L^1(G)$.

2. Let G be compact, and $\mathrm{Prob}(G) = \{\mu \in M_+(G) : \mu(G) = 1\}$.

   (a) Suppose $G = \prod_{i \in I} G_i$ (each G_i compact) where G_i is non-trivial for infinitely many i. For each i let m_i denote the Haar measure on each G_i, normalized so $m_i(G_i) = 1$, and $\mu_i = m_i \times \delta_{e_{\hat{\imath}}}$ where $e_{\hat{\imath}}$ is the identity in $G_{\hat{\imath}} = \prod_{j \in I \setminus \{i\}} G_j$. For each finite subset F of I consider the the convolution product $\mu_F = \ast_{i \in F} \mu_i$. (Why does product order not matter here?) Show that each μ_F is singular with respect to the normalised Haar measure m. Furthermore, the net $(\mu_F)_F$, indexed over the directed set of increasing finite subsets, converges in the weak* topology to m.

(b) Given μ in $M(G)$, let

$$\mathrm{supp}(\mu) = \{x \in G : |\mu|(U) > 0 \text{ for each open neighbourhood of } x\}.$$

Show that $\mathrm{supp}(\mu) = \bigcap\{F : F \text{ is closed, with } |\mu|(G\backslash F) = 0\}$, and hence is the smallest closed set supporting μ. [Compactness of G is not special here.]

(c) Show that if $x \in G$, then $\overline{\{x^n : n \in \mathbb{N}\}}$ is a subgroup of G.

(d) Let $\mu \in \mathrm{Prob}(G)$. Suppose the group $\langle \mathrm{supp}(\mu) \rangle$ is dense in G. Then the sequence of Cesaro averages of k-fold convolution products, $\frac{1}{n}\sum_{k=1}^{n} \mu^{*k}$, converges weak* to m. [Let $\bar{\mu}$ be any cluster point. First show that if $E \in \mathcal{B}(G)$ then $\mu_E * \bar{\mu} \leqslant \mu(E)\bar{\mu}$, then deduce that $\delta_x * \bar{\mu} \leqslant \bar{\mu}$ for any $x \in \mathrm{supp}(\mu)$.]

3. Let H be another locally compact group and $\beta : G \to H$ be a continuous homomorphism.

   (a) For μ in $M(G)$ show that $h \mapsto \int_G h \circ \beta \, d\mu$ on $C_0(H)$ defines a measure $\beta_M(\mu)$ in $M(H)$ with $\|\beta_M(\mu)\|_1 \leqslant \|\mu\|_1$, with equality holding if β is injective. Hence this defines a bounded map $\beta_M : M(G) \to M(H)$. Show moreover that β_M is an algebra homomorphism. [Warning: though $h \circ \beta \in C_b(G)$, it need not be in $C_0(G)$.]

   (b) Let N be a closed normal subgroup of G. Show that the map $T_N : C_c(G) \to C_c(G/N)$ defined for Weil's relation, extends to a surjective homomorphism $T_{N,L} : L^1(G) \to L^1(G/N)$. [Surjectivity is the hard part.]

   (c) Show that if β is open, i.e. $\beta(U)$ is open in H whenever U is open in G, then β_M induces a map $\beta_L : L^1(G) \to L^1(H)$. [It may be worthwhile to first consider the case where $H = G$ and β is an automorphism.]

4. Let H be a locally compact group and $\eta : H \to G$ be an injective continuous homomorphism.

   (a) Show that if H is σ-compact, then $\eta(H)$ is a Borel set.

   (b) Show, by way of an example, that (a) above may fail if we do not assume that H is σ-compact. [Hint: show that $\mathbb{R}$ admits a subgroup H with $\mathbb{R}/H$ countable.]

(c) Assume, hereafter, that

- $\eta(H)$ is Borel subset and normal subgroup of G, and
- each automorphism γ_x on H, given by $\eta(\gamma_x(y)) = x\eta(y)x^{-1}$ $(x \in G,\ y \in H)$, is continuous on H.

Let $\tau_{\eta(H)} \subseteq \mathcal{P}(G)$ be the topology with base

$$\{x\eta(U) : U \in \tau_H \text{ (i.e. } U \text{ is open in } H\text{) and } x \in G\}$$

Show that $\tau_{\eta(H)}$ is finer than the ambient topology τ_G on G, and that $(G, \tau_{\eta(H)})$ is a locally compact group.

(d) Let $M(G_{\eta(H)})$ be the measure algebra of $(G, \tau_{\eta(H)})$, and $\iota : (G, \tau_{\eta(H)}) \to (G, \tau_G)$ be the identity map. Show that

$$\iota_M\big(M(G_{\eta(H)})\big) = \ell^1\text{-}\bigoplus_{x \in T} \delta_x * \eta_M\big(M(H)\big)$$

where T is a transversal for cosets of $\eta(H)$ in G.

(e) Show that

$$I_{\eta(H)}(G) = \left\{\mu \in M(G) : \begin{array}{c}\mu(x\eta(K)) = 0 \text{ whenever } K \text{ is a} \\ \text{compact subset of } H \text{ and } x \in G\end{array}\right\}$$

is an ideal in $M(G)$. [You may wish to verify that each x in G determines a continuous automorphism α_x on H: $\eta(\alpha_x(y)) = x\eta(y)x^{-1}$.]

Examples to keep in mind, above. You should check them for your own edification

- If $H = \{e\}$ and $\eta : \{e\} \to G$ the only possible homomorphism, then $(G, \tau_{\eta(H)}) = G_d$, i.e. G with the discrete topology. Then $\iota_M\big(M(G_{\eta(H)})\big) = M_d(G) \cong \ell^1(G)$. [If H is countable and normal in G, the same conclusions hold.]

- If $G = H^2$ and $\eta : H \to G$ is given $\eta(x) = (x, e)$, then $(G, \tau_{\eta(H)}) = H \times H_d$. In this case, each element of $\iota_M\big(M(G_{\eta(H)})\big)$ is of the form $\nu = \sum_{x \in H} \mu_x \times \delta_x$ where $\|\nu\|_1 = \sum_{x \in H} \|\mu_x\|_1$ (and hence all but countably many of the measures μ_x in $M(H)$ are 0).

- (Irrational wind) Let $\eta : \mathbb{R} \to \mathbb{T}^2$ be given by $\eta(t) = (e^{i\xi t}, e^{it})$ where $\xi \in \mathbb{R}\backslash\mathbb{Q}$. Then $\eta(\mathbb{R})$ is Borel but not closed in $\mathbb{T}^2$. In this case $\mathbb{T}^2_{\eta(\mathbb{R})} \cong \mathbb{R} \times \mathbb{T}_d$, and $\iota_M\big(M(\mathbb{T}^2_{\eta(\mathbb{R})})\big)$ admits a description accordingly.