PMATH 833 (950), Winter 2018

Assignment #1 Due: February 1.

Unless otherwise stated, (G, τ) always denotes a Hausdorff locally compact group.

1. Show that (G, τ) is *complete* in the following sense: If (x_α) is a net in G which satisfies the property that for every V in τ such that $e \in V$, the is α_V such that $x_\alpha^{-1} x_\beta \in V$ for $\alpha, \beta \geq \alpha_V$, then there is x_0 in G such that $\lim_\alpha x_\alpha = x_0$.

   [An analogous statement holds with $x_\alpha x_\beta^{-1} \in V$ for $\alpha, \beta \geq \alpha_V$, as well.]

2. **(a)** Let $U \in \tau$ satisfy that $\overline{U}$ is compact. Prove that $\overline{U}$ is either finite or uncountable.

   (b) Deduce that the only Hausdorff topology σ on a countable group Γ which allows (Γ, σ) to be a locally compact group is the discrete topology.

   (c) Exhibit an example of a countable topological group which is not locally compact.

3. A *disconnection* for (G, τ) is any pair $\{U, V\} \subseteq \tau \setminus \{\varnothing\}$ for which $G = U \cup V$ and $U \cap V = \varnothing$. We say that (G, τ) is *connected* if no disconnections exists. We say that (G, τ) is *totally disconnected* if for any $x \neq y$ in G there is a disconnection $\{U_x, V_y\}$ for which $x \in U_x$ while $y \in V_y$.

   (a) Let $U \in \tau$ with $e \in U$. Show that $H = \bigcup_{n=1}^\infty U^n$ contains an open subgroup of G. Deduce that if G is connected, it is compactly generated, i.e. there is a compact set L for which the smallest subgroup containing L is all of G.

   (b) Show that if (G, τ) is totally disconnected, then every U in τ with $e \in U$ contains a compact W in τ with $e \in W$. [This can be done without recourse to the general fact that if a locally compact space is totally disconnected it is 0-dimensional.] [Hint: first suppose that $\overline{U}$ is compact and find V in τ so $e \in \overline{V} \subseteq U$.]

   (c) Suppose there is $W \in \tau$ with $e \in W$ and W itself is compact. Prove that W contains a compact open subgroup K of G. Deduce that (G, τ) is totally disconnected if and only if τ admits a base at e consisting of open subgroups. [Hint: for the first part, show that continuity of multiplication allows us to find neighbourhood V of e for which $VW \subseteq W$.]

(d) Deduce that that if (G, τ) is totally disconnected, and N is closed normal subgroup of G, then $(G/N, \tau_{G/N})$ (quotient topology) is totally disconnected.

(e) Show that if (G, τ) is totally disconnected and compact, then there is a base $\mathcal{N}$ for τ at e consisting of open normal subgroups. Deduce that G embeds in a product of finite groups, and that τ is metrizable only if $\mathcal{N}$ can be arranged to be countable. [Hint: show that if K is an open subgroup, then $\bigcap_{x \in G} xKx^{-1}$ may be realised as a finite intersection of conjugates of K.]

(e) Show that any closed subgroup Γ of $\mathrm{GL}_n(\mathbb{R})$, which is totally disconnected (in the relative topology) is necessarily discrete. Deduce that if (G, τ) is totally disconnected and compact, then any continuous homomorphism $\eta : G \to \mathrm{GL}_n(\mathbb{R})$ has finite range. [Hint: use linear algebra to study orbits $\{a^n\}_{n \in \mathbb{Z}}$ for $a \in \mathrm{GL}_n(\mathbb{R}) \setminus \{e\}$.]

4. Let (A, σ) be a locally compact group. We say that (A, σ) acts continuously on (G, τ) if for α in A, $x \mapsto \alpha(x)$ is an automorphism and the map $(x, \alpha) \mapsto \alpha(x) : G \times A \to G$ is $\tau \times \sigma - \tau$ continuous. Let m_G denote the left Haar measure on G.

   (a) Show that there is a continuous homomorphism $\delta : A \to (0, \infty)$ defined by $\delta(\alpha) m_G(E) = m_G(\alpha(E))$ for $E \in \mathcal{B}(G)$.

   (b) Define the semi-direct product of G by A by

   $$G \rtimes A = G \times A \text{ (as a set), with product } (x, \alpha)(y, \beta) = (x\alpha(y), \alpha\beta).$$

   Verify that $(G \rtimes A, \sigma \times \tau)$ is a locally compact group and that

   $$\int_{G \rtimes A} f \, dm = \int_G \int_A f(x, \alpha) \, \frac{dm_A(\alpha)}{\delta(\alpha)} \, dm_G(x), \;\; f \in \mathcal{C}_c(G \rtimes A)$$

   defines a left Haar integral on this group.

   (c) Compute formulas for both left and right Haar integrals on

   $$H = \left\{ \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} : a \in \mathrm{GL}_n(\mathbb{R}), x \in \mathbb{R}^n \text{ (column vectors)} \right\} \subset \mathrm{GL}_{n+1}(\mathbb{R}).$$

5. Let (G, τ) be a totally disconnected, compact and metrizable.

(a) Show that there is a sequential base $\mathcal{N} = \{N_k\}_{k=1}^{\infty}$ for τ at e, consisting of open normal subgroups such that

$$N_k \supseteq N_{k+1} \text{ for each } k.$$

(b) ("Riemann" sums) Given f in $\mathcal{C}(G)$, an $\mathcal{N}$-sequence is any sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{C}(G)$ such that

- for each k, $f_k(xn) = f_k(x)$ for each x in G and $n \in N_k$; and
- $\lim\limits_{k\to\infty} \|f - f_k\|_{\infty} = 0$.

Show that the limit

$$I(f) = \lim_{k\to\infty} \frac{1}{[G : N_k]} \sum_{xN_k \in G/N_k} f_k(x)$$

is independent of the choice of $\mathcal{N}$-sequence $(f_k)_{k=1}^{\infty}$ and defines an invariant integral on G.

(c) Let

$$G = \mathrm{GL}_2(\mathbb{O}_p) = \{a \in \mathrm{M}_2(\mathbb{O}_p) : \det a \in \mathbb{O}_p^{\times}\}.$$

Determine a base $\mathcal{N} = \{N_k\}_{k=1}^{\infty}$ for the topology at e as in (a) and compute the indices $[G : N_k]$.