The second dual of a C*-algebra

The main goal is to expose a “short” proof of the result of Z. Takeda, [*Proc. Japan Acad.* 30, (1954), 90–95], that the second dual of a C*-algebra is, in effect, the von Neumann algebra generated by its universal representation. Since our understood context is within a course in operator spaces, we will cheat by assuming the structure theorem for completely bounded maps into $\mathcal{B}(\mathcal{H})$, in particular as applied to bounded linear functionals.

Von Neumann algebras

Let us first recall that the *weak operator topology* (w.o.t.) on $\mathcal{B}(\mathcal{H})$ is the linear topology arising from $\mathcal{H}\otimes\mathcal{H}^*$. It is the coarsest topology which allows each functional $s\mapsto\langle s\xi|\eta\rangle$, where $\xi,\eta\in\mathcal{H}$, to be continuous. A subalgebra $\mathcal{M}\subset\mathcal{B}(\mathcal{H})$ containing the identity is called a *von Neumann algebra* if $\mathcal{M}$ is self-adjoint and w.o.t.-closed. It is obvious that the w.o.t. is coarser than the norm topology, hence a von Neumann algebra is a fortiori a C*-algebra.

Observe that if $\mathcal{A}_0$ is any unital self-adjoint subalgebra of then $\overline{\mathcal{A}_0}^{wot}$ is a von Neumann algebra. Indeed, observe that if (a_α), (b_β) are nets in $\mathcal{A}$ converging to x, y, respectively then for any ξ,η in $\mathcal{H}$ we have

$$\langle x^*\xi|\eta\rangle=\lim_\alpha\langle\xi|a_\alpha\eta\rangle=\lim_\alpha\langle a_\alpha^*\xi|\eta\rangle$$
$$\langle xy\xi|\eta\rangle=\lim_\beta\langle b_\beta\xi|x^*\eta\rangle=\lim_\beta\lim_\alpha\langle a_\alpha b_\beta\xi|\eta\rangle$$

and hence $x^*,ay\in\overline{\mathcal{A}_0}^{wot}$.

We may weaken the assumption the $\mathcal{A}_0$ is unital: we may simply assume that $\mathcal{A}_0$ is non-degenerately acting, i.e. $\underline{\text{span}\mathcal{A}_0\mathcal{H}}$ is dense in $\mathcal{H}$. This does require a bit of technology. The norm closure $\mathcal{A}=\overline{\mathcal{A}_0}$ is a C*-algebra hence contains a bounded approximate identity.

[Consider the set $\mathcal{F}$ of all finite subsets of hermitian elements of $\mathcal{A}$ directed by containment. Then for F in $\mathcal{F}$ we let

$$e_F=|F|\sum_{a\in F}a^2\left(I+|F|\sum_{a\in F}a^2\right)^{-1}\in\mathcal{A}_+.$$

Then for a in $\mathcal{A}$ we find for F containing $(aa^*)^{1/2}$ that

$$(I-e_F)aa^*(I-e_F)\leq\|a\|^2\left(I+|F|\sum_{a\in F}a^2\right)^{-2}\leq\frac{1}{|F|^2}I$$

and hence $\|a - e_F a\|^2 = \|(I - e_F)aa^*(I - e_F)\| \leq \frac{1}{|F|^2}$.]

Now if $\xi \in \mathcal{H}$, as above, approximate ξ by an element of span$\mathcal{A}_0\mathcal{H}$, and see that $e_F\xi$ gets arbitrarily close to ξ in norm. It follows that w.o.t.-$\lim_F e_F = I$.

Approximation by bounded nets

Let $\mathcal{A}_0$ be a non-degenerately acting self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Let us establish the significant fact that any element of $\overline{\mathcal{A}_0}^{wot}$ can be approximated by a net of bounded elements from within $\mathcal{A}_0$. This does not follow form general functional analytic principles, and requires a functional calculus technique which is Kaplansky's density theorem. This, in turn, requires a new topology on $\mathcal{B}(\mathcal{H})$.

On $\mathcal{B}(\mathcal{H})$ we define the *strong operator topology* (s.o.t.) as the initial topology generated by the functionals $s \mapsto \|s\xi\|$, $\xi \in \mathcal{H}$. Observe that a neighbourhood basis for this topology is formed by the inverse images of open sets of the functionals

$$s \mapsto \sum_{i=1}^{n} \|s\xi_i\|^2 \text{ for } \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathcal{H}^n,\, n \in \mathbb{N}.$$

Indeed, $\bigcap_{i=1}^n \{s \in \mathcal{B}(\mathcal{H}) : \|s\xi_i - s_0\xi_i\| < \epsilon\} \supseteq \left\{s \in \mathcal{B}(\mathcal{H}) : \sum_{i=1}^n \|s\xi_i - s_0\xi_i\|^2 < \epsilon^2\right\}$. Let us also remark that w.o.t. is the coarsest topology which allows each of the following functionals to be continuous:

$$s \mapsto \sum_{i=1}^{n} \langle s\xi_i | \eta_i \rangle \text{ for } \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathcal{H}^n,\, n \in \mathbb{N}.$$

Lemma. *Let $\mathcal{C}$ be a convex set in $\mathcal{B}(\mathcal{H})$. Then the s.o.t. and w.o.t.-closures of $\mathcal{C}$ coincide, i.e.*

$$\overline{\mathcal{C}}^{sot} = \overline{\mathcal{C}}^{wot}.$$

Proof. For ξ in the Hilbert space $\mathcal{H}^n$ let

$$\mathcal{C}_\xi = \{(n \cdot s)\xi : s \in \mathcal{C}\}$$

which is convex in $\mathcal{H}^n$. Here $n \cdot s$ is the diagonal ampliation of s. For s in $\mathcal{B}(\mathcal{H})$ observe that

$$\begin{aligned} &(n \cdot s)\xi \in \overline{\mathcal{C}_\xi}^{\|\cdot\|} \text{ for all } \xi \in \mathcal{H}^n \quad \Leftrightarrow \quad s \in \overline{\mathcal{C}}^{sot} \\ \text{and} \quad &(n \cdot s)\xi \in \overline{\mathcal{C}_\xi}^{w} \text{ for all } \xi \in \mathcal{H}^n \quad \Leftrightarrow \quad s \in \overline{\mathcal{C}}^{wot}. \end{aligned}$$

For example, $(n \cdot s)\xi \in \overline{\mathcal{C}_\xi}^{w}$, if and only if there is a net $(s_\alpha) \subset \mathcal{C}$ such that for any η in $\mathcal{H}^n$ we have $\langle (n \cdot s_\alpha)\xi | \eta \rangle$ converges in α to $\langle (n \cdot s)\xi | \eta \rangle = \sum_{i=1}^{n} \langle (n \cdot s)\xi_i | \eta_i \rangle$.

However, the Hahn-Banach theorem tells us for each ξ that

$$\overline{\mathcal{C}_\xi}^{\|\cdot\|} = \overline{\mathcal{C}_\xi}^{w}.$$

Hence the result follows. $\square$

Kaplansky's Density Theorem. *Let $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H})$ be a non-degenerately acting self-adjont subalgebra. Then the unit ball $\mathrm{B}(\mathcal{A})$ is weak*-dense in $\mathrm{B}(\overline{\mathcal{A}}^{w*})$.*

Proof. We first observe that the last lemma provides that $\overline{\mathcal{A}}^{wot} = \overline{\mathcal{A}}^{sot}$. Further, if $\mathcal{A}_{0,h}$ denotes the real vector space of hermitian elements in $\mathcal{A}_0$, then $\overline{\mathcal{A}_{0,h}}^{wot}$ is the set of hermitian elements in $\overline{\mathcal{A}_0}^{wot}$. Indeed, since involution $s \mapsto s^*$ is w.o.t.-w.o.t. continuous, a net $(s_\alpha) \subset \mathcal{A}$ converging w.o.t. to $s = s^*$ has that $(\mathrm{Re} s_\alpha)$ converges w.o.t. to s and $(\mathrm{Im} s_\alpha)$ converges w.o.t. to 0. Combining with the lemma above, we get $\overline{\mathcal{A}_{0,h}}^{sot} = \overline{\mathcal{A}_{0,h}}^{wot} = (\overline{\mathcal{A}_0}^{wot})_h$.

Consider $f : \mathbb{R} \to [-1, 1]$ and $g : [-1, 1] \to \mathbb{R}$ given by

$$f(x) = \frac{2x}{1 + x^2} \quad \text{and} \quad g(y) = \frac{y}{1 + \sqrt{1 - y^2}}.$$

Then $f \circ g = \mathrm{id}_{[-1,1]}$. Notice that for $s, t \in \mathcal{B}(\mathcal{H})_h$ that

$$2[f(s) - f(t)] = 4(1 + s^2)^{-1}(s - t)(1 + t^2)^{-1} - f(s)(s - t)f(t)$$

Now if s.o.t.-$\lim_\alpha s_\alpha = s$ in $\mathcal{B}(\mathcal{H})_h$, then since $\|(1 + s_\alpha^2)^{-1}\| \leq 1$ and $\|f(s_\alpha)\| \leq 1$ we have for $\xi \in \mathcal{H}$ that

$$\|[f(s_\alpha) - f(s)]\xi\| \leq 4 \left\|(s_\alpha - s)(1 + s^2)^{-1}\xi\right\| + \|(s_\alpha - s)f(s)\xi\| \stackrel{\alpha}{\to} 0$$

i.e. s.o.t.-$\lim_\alpha f(s_\alpha) = f(s)$. Now suppose $s \in \mathrm{B}(\overline{\mathcal{A}}^{sot}) \cap \mathcal{B}(\mathcal{H})_h$. Then $g(s) \in \mathrm{B}(\overline{\mathcal{A}_0}^{sot}) \cap \mathcal{B}(\mathcal{H})_h$ too, since $\overline{\mathcal{A}_0}^{sot}$ is a C*-algebra. Hence there is a net $(s_\alpha) \subset \mathcal{A}_o$ for which s.o.t.-$\lim_\alpha s_\alpha = g(s)$. But then s.o.t.-$\lim_\alpha f(s_\alpha) = f \circ g(s) = s$ and each $\|f(s_\alpha)\| \leq 1$. Note w.o.t.-$\lim_\alpha f(s_\alpha) = s$ too.

Now let $t \in \mathrm{B}(\overline{\mathcal{A}}^{wot})$ and consider the contractive hermitian element

$$s = \begin{bmatrix} 0 & t^* \\ t & 0 \end{bmatrix} \text{ in } \mathrm{M}_2(\overline{\mathcal{A}}^{w*}) = \overline{\mathrm{M}_2(\mathcal{A})}^{wot} \subset \mathrm{M}_2(\mathcal{B}(\mathcal{H})).$$

There is a net $(s_\alpha) \subset \mathrm{B}(\mathrm{M}_2(\mathcal{A})) \cap \mathrm{M}_2(\mathcal{A})_h$ such that w.o.t.-$\lim_\alpha s_\alpha = s$. Then $(s_{\alpha,21}) \subset \mathrm{B}(\mathcal{A})$ converges w.o.t. to t. □

We recall that the *weak*-toplogy* on $\mathcal{B}(\mathcal{H})$ is that arising from the predual $\mathcal{H} \otimes^\gamma \mathcal{H}^*$.

Corollary. *Given a non-degenerateley acting self adjoint subalgebra $\mathcal{A}_0$ of $\mathcal{B}(\mathcal{H})$, we have that its w.o.t. and weak*-closures coincide, i.e.*

$$\overline{\mathcal{A}_0}^{wot} = \overline{\mathcal{A}_0}^{w*}.$$

Moreover, each elements in $\overline{\mathcal{A}_0}^{w*}$ may be weak-approximated by a net of bounded elements.*

Proof. It clearly suffices to show that $\overline{\mathcal{A}_0}^{wot} \subseteq \overline{\mathcal{A}_0}^{w*}$. Since the w.o.t. is coarser that weak*-topology and is Hausdorff, by Banach-Alaoglu the topologies coincide on bounded sets. Hence any element of $\overline{\mathcal{A}_0}^{wot}$, is the w.o.t. limit of a net of bounded elements, hence the weak*-limit of such a net, and thus in the weak*-closure. □

Though we do not require the following result, we are so close it that it would be a shame not to do it. If $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ let the *commutant* of $\mathcal{V}$ be given by $\mathcal{V}' = \{x \in \mathcal{B}(\mathcal{H}) : xv = vx \text{ for all } v \text{ in } \mathcal{V}\}$ and its double commutant by $\mathcal{V}$" $= (\mathcal{V}')'$. Any commutant is easily checked to be a weak*-closed algebra. Furthermore, any commutant of a self-adjoint set is easily checked to be self-adjoint. Finally, it is clear that $\mathcal{V} \subseteq \mathcal{V}''$.

Von Neumann's Double Commutant Theorem. *Given a non-degenerateley acting self adjoint subalgebra $\mathcal{A}_0$ of $\mathcal{B}(\mathcal{H})$, we have that*

$$\overline{\mathcal{A}_0}^{sot} = \overline{\mathcal{A}_0}^{wot} = \mathcal{A}_0''.$$

In particular, these sets also coincide with $\overline{\mathcal{A}_0}^{w*}$.

Proof. From comments above, the only inclusion which needs to be checked is $\mathcal{A}_0'' \subseteq \overline{\mathcal{A}_0}^{sot}$. Let $\mathcal{D} = \{n{\cdot}a : a \in \mathcal{A}_0\} \subset \mathcal{B}(\mathcal{H}^n)$, which is a non-degenerately acting algebra on $\mathcal{H}^n$. It is clear that $\mathcal{D}''\tilde{\supseteq}\mathrm{M}_n(\mathcal{A}_0'')$ in $\mathcal{B}(\mathcal{H}^n) \cong \mathrm{M}_n(\mathcal{B}(\mathcal{H}))$. Fix $\xi \in \mathcal{H}^n$, let $\mathcal{K} = \overline{\mathrm{span}}\mathcal{D}\xi$ and then let p denote the orthogonal projection onto $\mathcal{K}$. Notice that $dp = pdp$ for $d \in \mathcal{D}$, and, as $\mathcal{D}$ is self-adjoint, $pd = (d^*p)^* = pdp$ too. Hence $p \in \mathcal{D}'$ so for $x \in \mathcal{D}''$, $xp = px$ too, and we thus see that $x\xi \in \mathcal{K}$. But hence, by definition of $\mathcal{K}$, for any $\epsilon > 0$ there there is $d \in \mathcal{D}$ so $\|d\xi - x\xi\| < \epsilon$. Letting $x = n \cdot t$ for some t in $\mathcal{A}_0''$ and writing $d = n \cdot a$, we obtain that

$$\sum_{i=1}^{n} \|(a-t)\xi_i\|^2 < \epsilon^2$$

which is what we wished to show. □

The dual and second dual of a unital C*-algebra

Given a unital C*-algebra $\mathcal{A}$, let $\mathcal{S}(\mathcal{A})$ denote its state space, and for each f iin $\mathcal{S}(\mathcal{A})$, $(\pi_f, \mathcal{H}_f, \xi_f)$ its Gelfand-Naimark triple. We let

$$\varpi = \bigoplus_{f \in \mathcal{S}(\mathcal{A})} \infty \cdot \pi_f \text{ on } \mathcal{H}_\varpi = \ell^2\text{-}\bigoplus_{f \in \mathcal{S}(\mathcal{A})} \mathcal{H}_f^{(\infty)}$$

where each $\infty \cdot \pi_f$ is the $\mathbb{N}$-ampliation of π_f on $\mathcal{H}_f^{(\infty)}$. This ampliation is not strictly necessary, but allows an aspect of the proof of part (i) of the theorem below to be seen more easily.

Theorem. *Let* $\mathcal{A}$ *be a unital C*-algebra.*
(i) *Each element of the dual* $\mathcal{A}^*$ *is of the form* $\varphi = \langle \varpi(\cdot)\xi|\eta\rangle$ *for a pair of vectors* ξ, η *in* $\mathcal{H}_\varpi$ *with* $\|\xi\|\,\|\eta\| = \|\varphi\|$.
(ii) *The second dual* $\mathcal{A}^{**}$ *is isometrically isomorphic with* $\overline{\varpi(\mathcal{A})}^{w*} \subset \mathcal{B}(\mathcal{H}_\varpi)$.

With the double commutation theorem in mind, one may wish to replace $\overline{\varpi(\mathcal{A})}^{w*}$ with $\varpi(\mathcal{A})''$. Furthermore, this result holds for a non-unital C*-algebra $\mathcal{A}$. We must keep in mind the fact that each cyclic representation of $\mathcal{A}$ is non-degereate (this was not shown in the other handout, but is true) and ϖ, being the direct sum of such, enjoys the same property.

Proof. **(i)** The structure theorem for completely bounded maps tells us that each element of $\mathcal{A}^*$ if of the form $\varphi = \langle \pi(\cdot)\xi'|\eta'\rangle$ for some representation π of $\mathcal{A}$ and $\xi'|\eta'$ in $\mathcal{H}_\pi$ with $\|\xi'\|\,\|\eta'\| = \|\varphi\|$. We consider a maximal family of mutually orthogonal projections $\{p_\alpha\}_{\alpha\in A}$ where each $p_\alpha\mathcal{H}_\pi$ is a cyclic subspace for π. Notice that $\sum_{\alpha\in A}\|p_\alpha\xi'\|^2 = \|\xi'\|^2 < \infty$, so there is a sequence of distinct indices $\{\alpha_i\}_{i=1}^\infty$ for which $\xi_i' = p_{\alpha_i}\xi' \neq 0$. Write, also, $\eta_i' = p_{\alpha_i}\eta'$ and $\pi_i = p_{\alpha_i}\pi(\cdot)|_{p_{\alpha_i}\mathcal{H}}$. Observe that $\sum_{i=1}^\infty \|\eta_i'\|^2 \leq \|\eta'\|^2$. Now for each i, there is f in $\mathcal{S}(\mathcal{A})$ for which $\pi_i \cong \pi_f$ (unitary equivalence), i.e. consider $f = \langle \pi_i(\cdot)\zeta_i|\zeta_i\rangle$, where ζ_i is a norm 1 cyclic vector. Thus $\langle \pi_i(\cdot)\xi_i'|\eta_i'\rangle = \langle \pi_f(\cdot)\xi_i|\eta_i\rangle$ for some ξ_i, η_i in $\mathcal{H}_f$ with $\|\xi_i\|\,\|\eta_i\| = \|\xi_i'\|\,\|\eta_i'\|$. Then we have

$$\begin{aligned}\varphi = \langle \pi(\cdot)\xi'|\eta'\rangle &\overset{(\dagger)}{=} \sum_{i=1}^\infty \langle \pi_i(\cdot)\xi_i'|\eta_i'\rangle = \sum_{i=1}^\infty \langle \pi_f(\cdot)\xi_i|\eta_i\rangle \\ &= \sum_{f\in\mathcal{S}(\mathcal{A})}\ \sum_{i:\pi_i\cong\pi_f} \langle \pi_f(\cdot)\xi_i|\eta_i\rangle = \langle \varpi(\cdot)\xi|\eta\rangle\end{aligned}$$

where we let

$$\xi = \big((\xi_i)_{\pi_i\cong\pi_f}\big)_{f\in\mathcal{S}(\mathcal{A})}, \eta = \big((\eta_i)_{\pi_i\cong\pi_f}\big)_{f\in\mathcal{S}(\mathcal{A})} \in \mathcal{H}_\varpi = \ell^2\text{-}\bigoplus_{f\in\mathcal{S}(\mathcal{A})} \mathcal{H}_f^{(\infty)}.$$

We observe that the series at $(\dagger)$ makes sense since each $\|\langle \pi_i(\cdot)\xi_i'|\eta_i'\rangle\| \leq \|\xi_i'\|\,\|\eta_i'\|$, by Cauchy-Schwarz, and another application of Cauchy-Schwarz gives

$$\sum_{i=1}^\infty \|\xi_i'\|\,\|\eta_i'\| \leq \left(\sum_{i=1}^\infty \|\xi_i'\|^2\right)^{1/2}\left(\sum_{i=1}^\infty \|\eta_i'\|^2\right)^{1/2} \leq \|\xi'\|\,\|\eta'\| < \infty.$$

Dual spaces are complete, so absolutely convergent series converge. Finally, computations just as above show that $\|\xi\|\,\|\eta\| \leq \|\xi'\|\,\|\eta'\| = \|\varphi\|$. But, of course, the converse inequality is automatic.

(ii) For each $A \in \mathcal{A}^{**}$ use Goldstine's theorem to find a net $(a_\alpha) \subset \mathcal{A}$ (with each $\|a_\alpha\| \leq \|A\|$) so w*-$\lim_\alpha \hat{a}_\alpha = A$. Here $\hat{a}$ denotes a as an evaluation functional on $\mathcal{A}^*$. Then define $\tilde{\varpi}(A) \in \mathcal{B}(\mathcal{H}_\varpi)$ by

$$\langle \tilde{\varpi}(A)\xi|\eta\rangle = \lim_\alpha \langle \varpi(a_\alpha)\xi|\eta\rangle \text{ for } \xi, \eta \in \mathcal{H}_\varpi.$$

It is easy to see that $\tilde{\varpi}(A)$ is well-defined, i.e. independent of choice of net, that $A \mapsto \tilde{\varpi}(A)$ is linear and that $\tilde{\varpi}(A) = \text{w*-}\lim_\alpha \varpi(a_\alpha) \in \overline{\varpi(\mathcal{A})}^{w*}$.

Furthermore, simply noting that

$$\big\langle A, \langle \varpi(\cdot)\xi|\eta\rangle\big\rangle = \langle\tilde{\varpi}(A)\xi|\eta\rangle$$

for each $\langle\varpi(\cdot)\xi|\eta\rangle$ in $\mathcal{A}^*$ – whose form is guaranteed by (i), above – we see that $\|\tilde{\varpi}(A)\| = \|A\|$. Finally, $\tilde{\varpi} : \mathcal{A}^{**} \to \overline{\varpi(\mathcal{A})}^{w*}$ is surjective. Indeed, given x in $\overline{\varpi(\mathcal{A})}^{w*}$, find a net $(a_\alpha) \subset \mathcal{A}$ so $x = \text{w*-}\lim_\alpha \varpi(a_\alpha)$. By the corollary to Kaplansky's density theorem we may suppose (a_α) is bounded. Hence by there is a subnet $(a_{\alpha(\beta)})$ so that $A = \text{w*-}\lim_\beta \hat{a}_{\alpha(\beta)}$ exists in $\mathcal{A}^{**}$. It follows that $x = \tilde{\varpi}(A)$. □

Necessity of Kaplansky's Density Theorem

Let us observe that the bounded net at the end of the proof requires something like Kaplansky's density theorem, and cannot be deduced from more general functional analytic principles. The following is motivated by an example of N. Ozawa on mathoverflow [`questions/102328/`].

Let for $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $(p_i)_{i=1}^\infty$ be an orthogonal sequence of infinite dimensional projections on $\mathcal{H}$, e_i a unit vector in $p_i\mathcal{H}$ for each i Fix a state ω_i on $\mathcal{B}(p_i\mathcal{H})$ for which $\omega_i(p_i) = 1$ and $\omega_i|_{\mathcal{K}(\mathcal{H})} = 0$. [Since $\operatorname{dist}(p_i, \mathcal{K}(\mathcal{H})) = 1$, this is a possible by Hahn-Banach theorem.] Then set

$$\mathcal{V} = \bigcap_{i=1}^{\infty} \ker(2^i\langle\cdot e_i|e_i\rangle - \omega_i) \subset \mathcal{B}(\mathcal{H}).$$

Hence if $v \in \mathrm{B}(\mathcal{V})$ then $|\langle ve_i|e_i\rangle| = \frac{1}{2^i}|\omega(v)| \leq \frac{1}{2^i}$. Then $\mathcal{V}$ is a self-adjoint subspace which contains a full matrix unit in for each space $p_i\mathcal{B}(\mathcal{H})p_j$ $(i \neq j)$; and also almost a full matrix unit in each $p_i\mathcal{B}(\mathcal{H})p_i$ less the projection $e_i \otimes e_i^*$, but does contain the operator $(1-\frac{1}{2^i})e_i\otimes e_i^* - p_i$. Check that $\mathcal{V}$ is weak*-dense in $\mathcal{B}(\mathcal{H})$, but that $\overline{\mathrm{B}(\mathcal{V})}^{w*}$ contains no ball in $\mathcal{B}(\mathcal{H})$.

If one wishes for an example which is a unital subalgebra consider

$$\mathcal{T}_\mathcal{V} = \left\{ \begin{bmatrix} \alpha I & v \\ 0 & \alpha I \end{bmatrix} : \alpha \in \mathbb{C}, v \in \mathcal{V} \right\} \subset \mathrm{M}_2(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^2).$$

Here $\overline{\mathcal{T}_\mathcal{V}}^{w*} = \mathcal{T}_{\mathcal{B}(\mathcal{H})}$ (definition of the latter space should be evident) but again the weak*-closure of $\mathrm{B}(\mathcal{T}_\mathcal{V})$ contains no balls in $\mathcal{T}_{\mathcal{B}(\mathcal{H})}$.

Written by Nico Spronk, for use by students of PMath 822 at University of Waterloo.