PMATH 822, FALL 2013

Assignment #2 Due: November 15.

1. Given a group G, let $\mathbb{C}[G]$ denote its group algebra and $\mathrm{C}^*(G)$ it *universal C*-algebra* which is the completion of $\mathbb{C}[G]$ with respect to the norm

$$\left\|\sum_{s\in G}\alpha(s)s\right\|_u = \sup\left\{\left\|\sum_{s\in G}\alpha(s)\pi(s)\right\|_{\mathcal{B}(\mathcal{H})} : \begin{matrix}\pi\in\mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))\\ \mathcal{H}\text{ a Hilbert space}\end{matrix}\right\}$$

where $\mathcal{U}(\mathcal{H})$ is the unitary group on $\mathcal{H}$. We remark that if $x = \sum_{s\in G}\alpha(s)s \in \mathbb{C}[G]\setminus\{0\}$ then

$$0 < \left\|\sum_{s\in G}\alpha(s)\lambda(s)\right\|_{\mathcal{B}(\ell^2(G))} \leq \|x\|_u \leq \sum_{s\in G}|\alpha(s)| < \infty$$

where $\lambda : G \to \mathcal{U}(\ell^2(G))$ is the left regular representation given by $\lambda(s)f(t) = f(s^{-1}t)$. Hence $\|\cdot\|_u$ is a norm on $\mathbb{C}[G]$, and it is straightforward to check that $\|x^*x\|_u = \|x\|_u^2$, so it is a C*-norm.

(a) Let F_∞ be the free group on generators $\{s_i\}_{i\in\mathbb{N}}$. Show that the map

$$f \mapsto \sum_{i=1}^{\infty} f(i)s_i : \max\ell^1(\mathbb{N}) \to \mathrm{C}^*(\mathrm{F}_\infty)$$

is a complete isometry.

[Hint. Any contraction a in $\mathcal{B}(\mathcal{H})$ is a corner of a unitary

$$\begin{bmatrix} a & -(1-aa^*)^{1/2} \\ (1-a^*a)^{1/2} & a^* \end{bmatrix} \in \mathcal{B}(\mathcal{H}^2).$$

The free group has a natural universal property.]

(b) Let $\mathbb{Z}^{\oplus\infty}$ denote the direct sum of countably many copies of the free abelian group $\mathbb{Z}$. Let $\{e_i\}_{i\in\mathbb{N}}$ be the usual set of generators of $\mathbb{Z}^{\oplus\infty}$. Show that the map

$$f \mapsto \sum_{i=1}^{\infty} f(i)e_i : \min\ell^1(\mathbb{N}) \to \mathrm{C}^*(\mathbb{Z}^{\oplus\infty})$$

is a complete isometry.

(c) Deduce that $\min\ell_2^1 = \max\ell_2^1$ completely isometrically.

2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_C = \mathcal{B}(\mathbb{C}, \mathcal{H})$ and $\mathcal{H}_R = \mathcal{B}(\mathcal{H}^*, \mathbb{C})$ (†) the denote the column and row quantizations. [The choice of using $\mathcal{H}^*$ at (†) facilitates results such as (b), below. It also allows us to assign the row operator space structure is a coordinate-free manner.]

   (a) Show that $\mathcal{CB}(\mathcal{H}_C, \mathcal{H}'_C) = \mathcal{B}(\mathcal{H}, \mathcal{H}')$, completely isometrically. [I.e. $\mathcal{CB}(\mathcal{H}_C, \mathrm{M}_n(\mathcal{H}'_C)) = \mathcal{B}(\mathcal{H}^n, (\mathcal{H}')^n)$, isometrically.]

   (b) Deduce that $(\mathcal{H}_C)^* \cong (\mathcal{H}^*)_R$, completely isometrically, and furthermore that $\mathcal{CB}(\mathcal{H}_R, \mathcal{H}'_R) = \mathcal{B}(\mathcal{H}'^*, \mathcal{H}^*)$, completely isometrically.

   Note that $(\mathcal{H}^*)_R$ denotes the conjugate space $\mathcal{H}^*$ with row structure; whereas $(\mathcal{H}_R)^* = (\mathcal{H}^*)_C$, as follows from (b).

   (c) Deduce the completely isometric identifications:

$$\mathcal{H}_C \hat{\otimes} (\mathcal{H}^*)_R \cong \mathcal{T}(\mathcal{H}) \quad \text{and} \quad \mathcal{H}_C \check{\otimes} (\mathcal{H}^*)_R \cong \mathcal{K}(\mathcal{H}).$$

3. Let $\mathrm{K}_\infty = \mathcal{K}(\ell^2(\mathbb{N}))$, $\mathrm{T}_\infty = \mathcal{T}(\ell^2(\mathbb{N}))$ and $\mathrm{M}_\infty = \mathcal{B}(\ell^2(\mathbb{N}))$.

   (a) Show for an operator space $\mathcal{V}$ that there is a completely isometric identification

$$(\mathcal{V} \otimes_\vee \mathrm{K}_\infty)^* \cong \mathcal{V}^* \hat{\otimes} \mathrm{T}_\infty.$$

   [In class we showed a isometric identifications $(\mathcal{V} \otimes_\vee \mathrm{M}_n)^* \cong \mathcal{V}^* \hat{\otimes} \mathrm{T}_n$. Why are these complete isometries?]

   (b) Show for an operator space $\mathcal{V}$ that there is a completely isometric identification

$$(\mathcal{V} \otimes_\wedge \mathrm{T}_\infty)^* \cong \mathcal{V}^* \bar{\otimes} \mathrm{M}_\infty.$$

   [Obviously, the idea is to show $\bar{\otimes} = \bar{\otimes}^F$, here.]

   (c) Deduce that for a linear map $T : \mathcal{V} \to \mathcal{W}$ ($\mathcal{W}$ another operator space) that the following are equivalent:

   (i) T is a complete pseudo-quotient map;
   (ii) $T \otimes \mathrm{id} : \mathcal{V} \otimes_\vee \mathrm{K}_\infty \to \mathcal{W} \otimes_\vee \mathrm{K}_\infty$ is a pseudo-quotient map; and
   (iii) $T \otimes \mathrm{id} : \mathcal{V} \otimes_\wedge \mathrm{T}_\infty \to \mathcal{W} \otimes_\wedge \mathrm{T}_\infty$ is a pseudo-quotient map.

4. **(a)** Given a complete operator space $\mathcal{V}$, let $\mathrm{K}_\infty(\mathcal{V}) = \mathcal{V}\check{\otimes}\mathrm{K}_\infty$. It is very convenient to think of elements of $\mathrm{K}_\infty(\mathcal{V})$ as $\mathbb{N}\times\mathbb{N}$-matrices with entries in $\mathcal{V}$.

   Show for complete operator spaces $\mathcal{V}$ and $\mathcal{W}$ that an arbitrary element u of $\mathrm{M}_n(\mathcal{V}\hat{\otimes}\mathcal{W})$ admits for any $\varepsilon > 0$ a factorization

$$u = \alpha(v \otimes w)\beta$$

   where

$$\alpha \in \mathrm{M}_{n,\infty^2}, v \in \mathrm{K}_\infty(\mathcal{V}), w \in \mathrm{K}_\infty(\mathcal{W}), \beta \in \mathrm{M}_{\infty^2,n}$$
$$\text{and } \|\alpha\|\,\|v\|\,\|w\|\,\|\beta\| < \|u\|_\wedge + \varepsilon.$$

   Here M_{n,∞^2} denotes the space of matrices representing bounded linear operators from $\ell^2(\mathbb{N}) \otimes^2 \ell^2(\mathbb{N})$ to ℓ^2_n.

   [Start by showing that space of elements admitting the desired factorization is itself a complete space with respect to the implied norm.]

   (b) Let for a complete operator space $\mathcal{V}$, $\mathrm{K}_{n,\infty}(\mathcal{V}) = \mathcal{V}\check{\otimes}\mathrm{M}_{n,\infty}$, and likewise define $\mathrm{K}_{\infty,n}$.

   Show for complete operator spaces $\mathcal{V}$ and $\mathcal{W}$ that an arbitrary element u of $\mathrm{M}_n(\mathcal{V} \otimes^h \mathcal{W})$ admits for any $\varepsilon > 0$ a decomposition

   $u = v \odot w$ where $v \in \mathrm{K}_{n,\infty}(\mathcal{V}), w \in \mathrm{K}_{\infty,n}(\mathcal{W})$ and $\|v\|\,\|w\| < \|u\|_h + \varepsilon$.

5. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ and operator space.

   (a) Show that there are completely isometric identifications

$$\mathcal{H}_C \otimes_h \mathcal{V} = \mathcal{H}_C \otimes_\vee \mathcal{V} \text{ and } \mathcal{V} \otimes_h \mathcal{H}_C = \mathcal{V} \otimes_\wedge \mathcal{H}_C.$$

   State the analagous results for $\mathcal{H}_R$.

   (b) Show that $\mathcal{CB}(\mathcal{H}_R, \mathcal{H}_C) = \mathcal{HS}(\mathcal{H})_C$, completely isometrically.

   (c) Deduce that the linear flip operator Σ on $\mathrm{K}_\infty \otimes_h \mathrm{K}_\infty$, given on elementary tensors by $\Sigma(k \otimes k') = k' \otimes k$, is unbounded.

6. Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces and

$$\Gamma_R(\mathcal{V}, \mathcal{W}^*) = \left\{ G : \mathcal{V} \to \mathcal{W}^* \;\middle|\; \begin{array}{c} G = T \circ S, S \in \mathcal{CB}(\mathcal{V}, \mathcal{H}_R) \\ T \in \mathcal{CB}(\mathcal{H}_R, \mathcal{W}^*), \mathcal{H} \text{ Hilbert space} \end{array} \right\}$$

Let $\|G\|_{\Gamma_R} = \inf\{\|S\|_{cb} \|T\|_{cb} : G = T \circ S \text{ as above}\}$

Show that $(\mathcal{V} \otimes_h \mathcal{W})^* \cong \Gamma_R(\mathcal{V}, \mathcal{W}^*)$ isometrically, via the dual pairing $\langle G, v \otimes w \rangle = \langle G(v), w \rangle$. Devise the matrix norms on $\Gamma_R(\mathcal{V}, \mathcal{W}^*)$ which allow this to be a complete isometry.

[First observe that $\mathcal{CB}(\mathcal{V}, \mathcal{H}_R) \cong \mathcal{V}^* \bar{\otimes} \mathcal{H}_R$ and $\mathcal{CB}(\mathcal{H}_R, \mathcal{W}^*) \cong (\mathcal{H}^*)_C \bar{\otimes} \mathcal{W}^*$. Think of these as long rows and long columns.]