PMATH 822, FALL 2013

Assignment #1 Due: October 11.

1. (Basic tensor product theory of normed spaces)

   Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$ and $\mathcal{Z}$ be normed vector spaces.

   For t in $\mathcal{X} \otimes \mathcal{Y}$ define the *projective tensor norm* by

$$\|t\|_\gamma = \inf\left\{ \sum_{i=1}^n \|x_i\|\,\|y_i\| : t = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

   and the *injective tensor norm* by

$$\|t\|_\lambda = \sup\left\{ \left| \sum_{i=1}^n f(x_i) g(y_i) \right| : t = \sum_{i=1}^n x_i \otimes y_i, f \in \mathrm{B}(\mathcal{X}^*) \text{ and } g \in \mathrm{B}(\mathcal{Y}^*) \right\}.$$

   [These are also known as the *greatest* and *least reasonable cross-norms*, hence the symbols γ and λ; but we shall not get into this discussion too deeply, though (b) provides adequate justification for γ.]

   (a) Show that $\|\cdot\|_\gamma$ and $\|\cdot\|_\lambda$ are norms on $\mathcal{X} \otimes \mathcal{Y}$.

   We let $\mathcal{X} \otimes_\alpha \mathcal{Y}$ denote the space $\mathcal{X} \otimes \mathcal{Y}$ with the norm $\|\cdot\|_\alpha$ with $\alpha = \gamma, \lambda$; and let $\mathcal{X} \otimes^\alpha \mathcal{Y}$ denote the completion of $\mathcal{X} \otimes_\alpha \mathcal{Y}$ with respect to the norm $\|\cdot\|_\alpha$.

   (b) (Universal property of $\otimes_\gamma$) Consider the space of bounded bilinear maps $\mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$, i.e. for $B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$, $\|B\| = \sup\{\|B(x, y)\| : x \in \mathrm{B}(\mathcal{X}), y \in \mathrm{B}(\mathcal{Y})\} < \infty$. Show that $B \mapsto T_B : \mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \to \mathcal{B}(\mathcal{X} \otimes_\gamma \mathcal{Y}, \mathcal{Z})$, defined on elementary tensors by $T_B(x \otimes y) = B(x, y)$, is a surjective isometry. Deduce that $(\mathcal{X} \otimes_\gamma \mathcal{Y})^* \cong \mathcal{B}(\mathcal{X}, \mathcal{Y}^*)$ isometrically.

   We call $\sigma(\mathcal{B}(\mathcal{X}, \mathcal{Y}^*), \mathcal{X} \otimes_\gamma \mathcal{Y})$ the *weak* operator topology* (or simply *weak operator topology* if $\mathcal{Y}$ is reflexive). We call $\sigma(\mathcal{B}(\mathcal{X}, \mathcal{Y}^*), \mathcal{X} \otimes^\gamma \mathcal{Y})$ the *weak* topology*. Notice that these topologies coincide on bounded sets. (Why?)

   (c) Embed $\mathcal{X} \otimes \mathcal{Y}^*$ into $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ by linearly extending the identifications $x \otimes f(y) = f(y)x$. Show that $\mathcal{X} \otimes_\lambda \mathcal{Y}^*$ is isometrically isomorphic with the family $\mathcal{F}(\mathcal{Y}, \mathcal{X})$ of finite rank operators in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$.

(d) Let $S : \mathcal{X} \to \mathcal{X}'$, $T : \mathcal{Y} \to \mathcal{Y}'$ be bounded linear maps. Show that $S \otimes T : \mathcal{X} \otimes \mathcal{Y} \to \mathcal{X}' \otimes \mathcal{Y}'$ is bounded when either projective or injective norms are simultaneously applied to each space. This extension is typically denoted $S \otimes T$.

Further show that if S and T are each isometries, then $S \otimes T : \mathcal{X} \otimes_\lambda \mathcal{Y} \to \mathcal{X}' \otimes_\lambda \mathcal{Y}'$ is an isometry (injective property). If S and T are each quotient maps (i.e. S is surjective and $\|x'\| = \inf\{\|x\| : Sx = x'\}$), then $S \otimes T : \mathcal{X} \otimes_\gamma \mathcal{Y} \to \mathcal{X}' \otimes_\gamma \mathcal{Y}'$ is also a quotient map (projective property).

(e) Use results from (b), (c) and (d) above to deduce that the embedding $\mathcal{X}^* \otimes_\lambda \mathcal{Y}^* \hookrightarrow (\mathcal{X} \otimes_\gamma \mathcal{Y})^*$, given on pairs of elementary tensors by $\langle f \otimes g, x \otimes y \rangle = f(x)g(y)$, is an isometry.

(f) Let X and Y be locally compact Haudorff spaces. Consider the map defined on elementary tensors $f \otimes g \mapsto f \times g$ from $\mathcal{C}_0(X) \otimes_\lambda \mathcal{C}_0(Y)$ to $\mathcal{C}_0(X \times Y)$, $f \times g(x, y) = f(x)g(y)$. Show that this is an isometry with dense range, hence $\mathcal{C}_0(X) \otimes^\lambda \mathcal{C}_0(Y) \cong \mathcal{C}_0(X \times Y)$ isometrically.

(g) Let X be a compact Haudorff space. Consider the map defined on elementary tensors $f \otimes y \mapsto f(\cdot)y$, from $\mathcal{C}(X) \otimes_\lambda \mathcal{Y}$ to $\mathcal{C}_0(X, \mathcal{Y}) = \{F : X \to \mathcal{Y} \mid F \text{ is continuous}\}$. Show that this map is an isometry with dense range, hence $\mathcal{C}(X) \otimes^\lambda \mathcal{Y} \cong \mathcal{C}(X, \mathcal{Y})$ isometrically.

(h) Let $(X, \mu), (Y, \nu)$ be measure spaces. Show that the map $f \otimes g \mapsto f \times g : \mathrm{L}^1(\mu) \otimes_\gamma \mathrm{L}^1(\nu) \to \mathrm{L}^1(\mu \times \nu)$, where $f \times g$ is defined $\mu \times \nu$-a.e., as above, is an isometry with dense range, hence $\mathrm{L}^1(\mu) \otimes^\gamma \mathrm{L}^1(\nu) \cong \mathrm{L}^1(\mu \times \nu)$ isometrically. [Hint. Consider the dense subspace $\mathrm{S}^1(\mu)$ of integrable simple functions.]

2. Now let $\mathcal{H}$ be a Hilbert space, $\mathcal{B} = \mathcal{B}(\mathcal{H})$, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ (compact operators), and $\mathcal{B}_+$ denote the cone of positve operators. If $x \in \mathcal{B}$, let $|a| = (a^* a)^{1/2}$.

   (a) Fix, for the moment, an o.n.b. $(e_i)_{i \in I}$ for $\mathcal{H}$. Let $\mathrm{Tr} : \mathcal{B}_+ \to [0, \infty]$ be given by $\mathrm{Tr}(a) = \sum_{i \in I} \langle a e_i | e_i \rangle = \sum_{i \in I} \left\| a^{1/2} e_i \right\|^2$. Show that this definition is independent of o.n.b. [Hint: Bessel & Tonelli.]

   (b) Show that for any t in $\mathcal{B}_+$ so $\mathrm{Tr}(t) < \infty$, that there is a sequence of finite rank $(t_n)_{n=1}^\infty \subset \mathcal{B}_+$ such that $t - t_n \in \mathcal{B}_+$ and $\lim_{n \to \infty} \mathrm{Tr}(t - t_n) = 0$. [Hint. Show first that $t \in \mathcal{K}_+$.]

Then for a in $\mathcal{B}$ let $\|a\|_1 = \mathrm{Tr}(|a|) \in [0,\infty]$. Let $\mathcal{T} = \{t \in \mathcal{B} : \|t\|_1 < \infty\}$ which denotes the space of *trace class* operators.

(c) For f in $\mathcal{K}^*$, show that t_f in $\mathcal{B}(\mathcal{H})$ defined by $\langle t_f \xi | \eta \rangle = f(\xi \otimes \eta^*)$ satisfies that $\|t_f\|_1 = \|f\|$. Moreover, deduce that the linear map from $\mathcal{H} \otimes_\gamma \mathcal{H}^*$ to $\mathcal{B}$, which identifies each elementary tensor $\xi \otimes \eta^*$ with the associated rank-one operator, is an isometry into a dense subspace of $\mathcal{T}$. Hence $\mathcal{H} \otimes^\gamma \mathcal{H}^* \cong \mathcal{T}$.

[Hint. Consider polar decomposition $t_f = u_f |t_f|$, and an o.n.b. $(e_i)_{i \in I}$ extending and o.n.b. for $\overline{t_f(\mathcal{H})}$ and the net $\left(\sum_{i \in F} e_i \otimes (u_f e_i)^*\right)_{F \subset I \text{ finite}}$. Observe that $\mathcal{K} \cong \mathcal{H} \otimes^\lambda \mathcal{H}^*$ (why?).]

(d) Verify that $\mathcal{T}$ is an ideal in $\mathcal{B}$ with

$$\|at\|_1 \leq \|a\| \, \|t\|_1 \text{ and } \|ta\|_1 \leq \|t\|_1 \, \|a\| .$$

Let now $\mathcal{HS} = \{h \in \mathcal{B} : \|h\|_2 = \mathrm{Tr}(h^*h)^{1/2} < \infty\}$, which denotes the space of *Hibert-Schmidt* operators. Note that the dual space $\mathcal{H}^*$ is a Hilbert space with inner product $\langle \xi^* | \eta^* \rangle_{\mathcal{H}^*} = \langle \eta | \xi \rangle_{\mathcal{H}}$ and scalar multiplication $\alpha \xi^* = (\bar{\alpha}\xi)^*$.

(e) Verify that $\mathcal{HS}$ is a Hilbert space which is isometrically isomorphic (i.e. unitarily equivalent) to $\mathcal{H} \otimes^2 \mathcal{H}^*$.

(f) Verify that $\mathcal{HS}$ is an ideal in $\mathcal{B}$ with

$$\|ah\|_2 \leq \|a\| \, \|h\|_2 \text{ and } \|ha\|_2 \leq \|h\|_2 \, \|a\| .$$

3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $(e_j)_{j \in J}$ be an orthonormal basis of $\mathcal{H}$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the C*-algebra of compact operators on $\mathcal{H}$. The collection of rank one partial isometries $\{e_i \otimes e_j^* : i, j \in J\}$ is called a *matrix unit.*

   (a) Let $\pi : \mathcal{K} \to \mathcal{B}(\mathcal{L})$ be a representation of $\mathcal{K}$ on another Hilbert space $\mathcal{L}$. Show that $\mathcal{L}$ admits a decomposition $\ell^2\text{-}\bigoplus_{j \in J} \mathcal{L}_j$, where the spaces $\mathcal{L}_j$ are pairwise isomorphic, and $\pi(e_i \otimes e_j^*)|_{\mathcal{L}_j}$ is an isometry whose image is $\mathcal{L}_i$. Thus deduce that there is a unitary $U : \mathcal{L} \to \mathcal{H}^\alpha$ such that $U\pi(\cdot)U^* = \alpha \cdot \mathrm{id}$, where $\alpha \cdot \mathrm{id} : \mathcal{K} \to \mathcal{B}(\mathcal{H}^\alpha)$ is the α-fold direct sum of the identity representation $\mathrm{id} : \mathcal{K} \hookrightarrow \mathcal{B}(\mathcal{H})$ and α is a cardinal.

[The net $\left(e_F = \sum_{i\in F} e_i \otimes e_i^*\right)_{F\subset J \text{ finite}}$ is an approximate identity for $\mathcal{K}$. We shall insist that representations are non-degenerate, a consequence of which is that w*-$\lim_{F\nearrow I} \pi(e_F) = I$. The weak* topology on $\mathcal{B}(\mathcal{H})$ is explained above.]

(b) Deduce that if $T : \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a completely bounded map, then there is a family of operators $\{a_i, b_i\}_{i\in I}$ (over some index set I) in $\mathcal{B}(\mathcal{H})$ such that $\sum_{i\in I} a_i a_i^*$ and $\sum_{i\in I} b_i^* b_i$ each converge in weak* topology and

$$Tk = \text{w*-}\sum_{i\in I} a_i k b_i \text{ and } \left\|\sum_{i\in I} a_i a_i^*\right\| \left\|\sum_{i\in I} b_i^* b_i\right\| = \|T\|_{cb}^2 .$$

(c) Show that if $T : M_n \to M_k$ is a completely positive map then there is a collection $\{a_i : i = 1, \dots, nk\}$ in $M_{k,n}$ such that

$$Tx = \sum_{i=1}^{nk} a_i x a_i^*.$$

If $k = n$, deduce necessary and sufficient conditions for $T1 = 1$ and for $\mathrm{Tr}(Tx) = \mathrm{Tr}(x)$, for each x in M_n, where Tr is the standard trace. Such maps are sometimes called *quantum channels*.

[Examine the proof of Stinespring to get the number nk.]