Introductory C*-algebra Theory

The following notes give the most basic results in C*-algebras. They have stolen liberally form W. Arveson's *An Invitation to C*-algebras*, where the reader will surely find the writing better and more concise.

Definition. A *C*-algebra* is a Banach space $\mathcal{A}$, equipped with and associative bilinear multiplcation $(a, b) \mapsto ab$ and a conjugate linear, anti-multiplicative involution $a \mapsto a^*$ (i.e. $(ab)^* = b^*a^*$, $(a^*)^* = a$), satisfying the following additional properties for each a and b:

(submultiplicativity)	$\|ab\| \leq \|a\| \, \|b\|$
(isomtric involution)	$\|a^*\| = \|a\|$
(C*-identity)	$\|a^*a\| = \|a\|^2$

A linear algebra homomorphism between C*-algebras $\pi : \mathcal{A} \to \mathcal{B}$ which is self-adjoint, i.e. $\pi(a^*) = \pi(a)^*$, is called a $*$-*homomorphism*. It is called a $*$-*isomorphsm* if it is bijective.

If we maintain only the axioms above which have nothing to do with involution, we obtain a *Banach algebra*.

Examples. **(i)** Let X be a locally compact Hausdorff space. Consider the space of continuous functions vanishing at infinity $\mathcal{C}_0(X)$ with pointwise addition and multiplication, involution $f^*(x) = \overline{f(x)}$ and uniform norm $\|\cdot\|_\infty$. This is easily checked to be a C*-algebra. Check that $\mathcal{C}_0(X)$ is unital (admits multiplicative identity) if and only if X is compact; we write $\mathcal{C}(X)$, in this case.

(ii) Consider bounded operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$, with pointwise addition, composition multiplication and involution defined by $\langle a^*\xi|\eta\rangle = \langle \xi|a\eta\rangle$, for ξ, η in $\mathcal{H}$. This too can, and should, be checked to be a C*-algebra. If $\mathcal{H} = \ell^2(n)$ is finite dimensional, identify $\mathcal{B}(\ell^2(n)) = \mathrm{M}_n$. Then $[a_{ij}]^* = [\overline{a_{ji}}]$.

(iii) Any closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ which is *self-adjoint*, i.e. $a^* \in \mathcal{A}$ whenever $a \in \mathcal{A}$, is a C*-algebra.

We say a C*-algebra $\mathcal{A} \neq \{0\}$ is *unital* if it admits a multiplicative identity 1. Observe that $1^*a = (a^*1)^* = a$, and similarly $a1^* = a$, hence 1^* functions as a multiplicative identity, so $1^* = 1^*1 = 1$. Moreover $\|1\| = \|1^*1\| = \|1\|^2$, and, since $1 \neq 0$, $\|1\| = 1$. Similarly check that any unitary element u, i.e. $u^*u = 1 = uu^*$, satisfies $\|u\| = 1$.

Unitization. *Any non-unital C*-algebra $\mathcal{A}$ embeds into a unital C*-algebra $\mathcal{A}_1$.*

Proof. Consider the space $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ with product $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$, involution $(a, \lambda)^* = (a^*, \bar{\lambda})$ and norm $\|(a, \lambda)\| = \sup\{\|ab + \lambda b\| : b \in \mathcal{A}, \|b\| \leq 1\}$. Check that this is a unital C*-algebra with idenity $1 = (0, 1)$. □

We define the *spectrum* of an element a in a unital $\mathcal{A}$ in the same way as for operators:

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ admits no inverse in } \mathcal{A}\}.$$

Notice that the usual arguements from functional analysis hold, and $\sigma(a)$ is always a non-empty compact subset of $\mathbb{C}$. In particular we get Beurling's spectral radius formula:

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

(In fact, these remain true in any unital Banach algebra.)

Iterations of the C*-identity show that for any hermitian element a in $\mathcal{A}$, i.e. $a = a^*$, $\left\|a^{2^n}\right\| = \|a\|^{2^n}$. It follows from the spectral radius formula that $r(a) = \|a\|$.

Commutative C*-algebras and normal funtional calculus

Let us observe that Example (i), above, is generic for commutative C*-algebras.

Commutative Gelfand-Naimark Theorem. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Then there is a locally compact space Ω for which $\mathcal{A} \cong \mathcal{C}_0(\Omega)$, isometically and $*$-isomorphically.*

Proof. We let

$$\Omega = \{\omega : \mathcal{A} \to \mathbb{C} \mid \omega \text{ is a continous algebra homomorphism}, \omega \neq 0\}.$$

Observe that each functional ω in Ω is contractive. Indeed, if a in $\mathcal{A}$ with $\|a\| < 1$, then $\omega(a)^n = \omega(a^n) \to 0$ as $n \to \infty$, so $|\omega(a)| < 1$. If (ω_α) is a net in Ω, converging weakly to ω in the dual space $\mathcal{A}^*$, then it is easy to check that $\omega \in \Omega_1 = \Omega \cup \{0\}$. Hence, by the Banach-Alaoglu theorem, Ω_1 is weak*-compact, hence Ω is weak* locally compact. Furthermore if $\mathcal{A}$ is unital, we see that $\omega(1) = 1$, hence $\omega \in \Omega$, in which case we see that Ω itself is weak*-compact.

Note that if $\mathcal{A}$ is non-unital, then each ω in Ω extends uniquely to $\mathcal{A}_1$. Also the functional $(a, \lambda) \mapsto \lambda$ uniquely extends the zero functional to a multiplicative linear functional of $\mathcal{A}_1$.

Thus let us, until said otherwise, assume that $\mathcal{A}$ is unital.

Let us show that $\sigma(a) = \{\omega(a) : \omega \in \Omega\}$ for a in $\mathcal{A}$. Indeed, it sufficies to show that any non-invertible a_0 admits an ω in Ω for which $\omega(a_0) = 0$. Now for such a_0, $\mathcal{A}a_0$ is a proper ideal in $\mathcal{A}$, hence by a standard Zorn's lemma argument for unital algebras, $\mathcal{A}a_0$ is contained in a maximal proper ideal $\mathcal{J}$. Notice that since any element within distance 1 of the identity 1 is invertible, $\text{dist}(1, \mathcal{J}) = 1$. Hence it follows that the closure $\overline{\mathcal{J}}$ is also a proper ideal, so $\mathcal{J}$ is itself closed. Furthermore $\mathcal{B} = \mathcal{A}/\mathcal{J}$ contains no proper ideals (since such would return an ideal in $\mathcal{A}$ containing $\mathcal{J}$), so all elements in $\mathcal{B}$ are invertible. Notice that $\mathcal{B}$ is a Banach algebra with norm $\|b + \mathcal{J}\| = \text{dist}(b, \mathcal{J})$ and identity $1 + \mathcal{J}$. If we had $b + \mathcal{J} \neq \lambda 1 + \mathcal{J}$ for any λ, then the spectrum of $b + \mathcal{J}$ in $\mathcal{B}$ would be empty, which is impossible. (The modest argument of this last sentence often goes by the name Banach-Mazur theorem.) Hence $\mathcal{B} = \mathbb{C}1 + \mathcal{J}$. In particular if we let $\omega(b)$ be defined by $b + \mathcal{J} = \omega(b)1 + \mathcal{J}$, then it is easy to check that $b \mapsto \omega(b)$ defines a continuous character on $\mathcal{A}$, which vanishes on our prescribed non-invertible a_0.

Let us show that each ω in Ω is self-adjoint, i.e. for each hermitian a in $\mathcal{A}$, $\omega(a) \in \mathbb{R}$. For such a we and t in $\mathbb{R}$ we have that $\exp(ita)$ — whose power series converges in $\mathcal{A}$ — is unitary: $\exp(ita)^* = \exp(-ita) = \exp(ita)^{-1}$. Thus we see that

$$
\begin{aligned}
1 &\geq |\omega(\exp(ita))| = |\exp(it\omega(a))| = |\exp(it\text{Re}\omega(a) - t\text{Im}\omega(a))| \\
&= |\exp(it\text{Re}\omega(a))||\exp(-t\text{Im}\omega(a))| = |\exp(-t\text{Im}\omega(a))|
\end{aligned}
$$

for all t, which shows that $\text{Im}\omega(a) = 0$. It thus follows that $\omega(a^*) = \overline{\omega(a)}$ (indeed write $a = \text{Re}a + i\text{Im}a = \frac{1}{2}(a + a^*) + \frac{1}{2i}(ia - ia^*)$).

We let Ω be endowed with the weak*-topology. Now define $\Gamma : \mathcal{A} \to \mathcal{C}(\Omega)$ by $\Gamma(a)(\omega) = \omega(a)$. The topology on Ω is defined exactly to facilitate that $\Gamma(\mathcal{A})$ indeed consists of continuous functions. It is trivial to check that Γ is a $*$-homomorphism; in particular $\Gamma(a^*) = \Gamma(a)^*$, thanks to the last paragraph. For hermitian a we have $\|a\| = r(a) = \|\Gamma(a)\|_\infty$, using facts gathered above.

Hence for general a we have $\|a\|^2 = \|a^*a\| = \|\Gamma(a)^*\Gamma(a)\|_\infty = \|\Gamma(a)\|^2$. Thus Γ is an isometry, and hence has closed range. Moreover, $\Gamma(\mathcal{A})$ is a point separating, conjugate-closed subalgebra of $\mathcal{C}(\Omega)$. Hence by Stone-Weierstrauss theorem, Γ is surjective.

Now if $\mathcal{A}$ is non-unital, we obtain isometric $*$-isomorphism $\Gamma : \mathcal{A}_1 \to \mathcal{C}(\Omega_1)$. Restricting Γ to $\mathcal{A}$ gives range $\mathcal{C}_0(\Omega)$. □

The set Ω, above, is called the *Gelfand spectrum* of $\mathcal{A}$, and frequently denoted $\widehat{\mathcal{A}}$. The map $\Gamma_{\mathcal{A}} = \Gamma : \mathcal{A} \to \mathcal{C}_0(\widehat{\mathcal{A}})$ is called the *Gelfand transform.* Notice that if $\mathcal{A}$ is separable, then by metrisation theorem $\Omega \subset \{f \in \mathcal{A}^* : \|f\| \leq 1\}$ is metrisable.

We should remark the following. It contains aspects of measure theory and topology which the reader should reveiw.

Proposition. *If X is a locally compact Hausdorff space, then $\Omega = \widehat{\mathcal{C}_0(X)} \cong X$, homeomorphically and $\Gamma_{\mathcal{C}_0(X)}$ is the identity operator.*

Proof. It is obvious that for each x in X, $\omega_x(f) = f(x)$ defines a continous multiplicative linear functional on $\mathcal{C}_0(X)$, and that $x \mapsto \omega_x$ is continuous (weak* topology used on $\mathcal{C}_0(X)$).

Let us see that all such characters arise accordingly. We use the Riesz representation theorem form measure theory, $\mathcal{C}_0(X)^* \cong \mathrm{M}(X)$ (complex measures on X). If $\omega \in \Omega$, there is a real signed measure μ such that $\omega(f) = \int_X f\, d\mu$. If supp$\mu$ contains two distinct points, we could use Tietze's extention theorem to find f, g in $\mathcal{C}_c(X)$ for which $\omega(f) \neq 0 \neq \omega(g)$, but $fg = 0$. This clearly violates that ω is mutiplicative. Hence suppμ is a singleton. This means that $\mu = \delta_x$, the Dirac measure at x, so $\omega = \omega_x$.

By Uryzohn's lemma $x \mapsto \omega_x$ is injective. In the case that X is not compact, the map extends uniquely to the one-point compactification X_∞, by $\infty \mapsto 0$. Hence we obtain a continuous bijection $X \cong \Omega$ if X is compact, and $X_\infty \cong \Omega \cup \{0\}$ otherwise. In either case we obtain a homeomorphism $X \cong \Omega$. □

Here is a significant reason for wishing to know the commutative Gelfand-Naimark theorm.

Normal Functional Calculus. *Let a be a normal element in a C^*-algebra $\mathcal{A}$, i.e. $a^*a = aa^*$ and let $\mathrm{C}^*_1(a)$ denote the closure of the algebra* $\mathrm{alg}(a, a^*, 1)$

generated by a, a^* and 1. Then there is a unique $*$-homomorphism $f \mapsto f(a) : \mathcal{C}(\sigma(a)) \to \mathrm{C}^*_1(a)$ which sends id *to a.*

Proof. Since $\sigma(a) = \{\omega(a) : \omega \in \Omega = \widehat{\mathrm{C}^*_1(a)}\}$, and elements of Ω are self-adjoint, the function $\Gamma(a) : \Omega \to \sigma(a)$ is continuous and bijective, hence a homeomorphism. Thus the map $b \mapsto \Gamma(b) \circ \Gamma(a)^{-1}$ is a $*$-isomorphism from $\mathrm{C}^*_1(a)$ onto $\mathcal{C}(\sigma(a)) = \{f \circ \Gamma(a)^{-1} : f \in \mathcal{C}(\Omega)\}$. It is evident that a is sent to id. Furthermore this map is uniquely defined on all elements of $\mathrm{alg}(a, a^*, 1)$, hence extends uniquely to $\mathrm{C}^*_1(a)$. □

Observe that the functional calculus, above, sends any polynomial function $z \mapsto p(z, \bar{z})$ on $\sigma(a)$ to $p(a, a^*)$. Hence the notation $f \mapsto f(a)$ is suggestive and warranted.

We observe that algebra homomorphisms $\omega : \mathcal{A} \to \mathbb{C}$ are automatically contractive, without assuming continuity as we did above. Indeed, suppose $\mathcal{A}$ is unital (or work within $\mathcal{A}_1$, as above). If $|\omega(a)| > \|a\|$ for some a, then $1 - \frac{1}{\omega(a)}a$ is invertible, and hence so too is $b = \omega(a)1 - a$. But then $\omega(b) = \omega(a)\omega(1) - \omega(a) = 0$, whence $\omega(1) = \omega(b)\omega(b^{-1}) = 0$, which is absurd. We aim to see that all $*$-homomorphisms enjoy the same property.

The following is a beautiful illustration of the power of functional calculus.

Automatic Continuity of $*$-Homomorphisms. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ a $*$-homomorphism. Then π is contractive. Furthermore, if π is injective, it is isometric.*

Proof. We first suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital and $\pi(1) = 1$. Then $\sigma(\pi(a)) \subseteq \sigma(a)$ for each a, i.e. π takes invertibles to invertibles, but may take non-invertibles to invertibles. Hence

$$\|a\|^2 = \|a^*a\| = r(a^*a) \leq r(\pi(a^*a)) = \|\pi(a)\|^2 .$$

Thus π is contractive.

Suppose π is injective. Let a be hermitian. Suppose f in $\mathcal{C}(\sigma(a))$ satisfies $f|_{\sigma(\pi(a))} = 0$. Approximate f uniformly by polynomials $(p_n)_{n=1}^\infty$ on $\sigma(a)$. Observe $p_n(\pi(a)) = \pi(p_n(a))$, and hence by continuity of functional calculus $f(\pi(a)) = \pi(f(a))$. Our assumptions on f provide $f(\pi(a)) = 0$, hence theinjectivity of π provides $f(a) = 0$. But then $f = 0$ on $\sigma(a)$, so $\sigma(\pi(a)) = \sigma(a)$.

Hence $r(a) = r(\pi(a))$ for any hermitian a, from which it follows from the C*-identity, as above, that π is isometric.

Suppose that $\mathcal{A}$ is not unital. Construct the unitisation $\mathcal{A}_1$ as above. By replacing $\mathcal{B}$ by $\mathcal{B}_1$, if necessary, we may suppose that $\mathcal{B}$ is unital. Define $\pi_1 : \mathcal{A}_1 \to \mathcal{B}$ by $\pi_1(a, \lambda) = \pi(a) + \lambda 1$. It is easy to see that π_1 is a $*$-homomorphism. $\pi = \pi_1|_{\mathcal{A}}$ is contractive. Furthermore π_1 is injective if π is injective, and hane isometric in that case, whence so too is π. $\square$

It is more delicate, but not too hard, to obtain that the quotient of any C*-algebra by a self-adjoint ideal is again a C*-algebra. Once that is known, we see that the induced map $\tilde{\pi} : \mathcal{A}/\ker\pi \to \mathcal{B}$ is an isometry, hence has closed range, which is necessarily a C*-algebra in $\mathcal{B}$. Details are left for the reader to look up.

States and representations

Our ultimate goal is to see that Example (iii), above, is generic. That is, for any C*-algebra $\mathcal{A}$, there is an injective $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Such a homomorphism will be called a *faithful representation*. Notice for comutative $\mathcal{A} \cong \mathcal{C}_0(\Omega)$, the map $J : \mathcal{A} \to \mathcal{B}(\ell^2(\Omega))$, $(Ja)\xi(\omega) = \Gamma_{\mathcal{A}}(a)(\omega)\xi(\omega)$ already suffices. In a certain sense, we will generalise this construction to a non-commutative setting.

A linear functional $f : \mathcal{A} \to \mathbb{C}$ is *positive* if $f(a^*a) \geq 0$ for each a in $\mathcal{A}$. We define a sesquilinear form on $\mathcal{A}$:

$$(a, b) \mapsto [a|b]_f = f(b^*a).$$

This form thus satisfies the *Cauchy-Schwarz inequality*:

$$\left|[a|b]_f\right| \leq [a|a]^{1/2}[b|b]^{1/2}.$$

Proposition. *A positive linear functional on a untal C*-algebra $\mathcal{A}$ is bounded with $\|f\| = f(1)$.*

Proof. Notice that $\|f\| \geq f(1)$ since $\|1\| = 1$.

First observe that if a is hermitian and $\|a\| \leq 1$, then each of $1 \pm a$ admits a hermitian square root, i.e. $b_\pm$ hermitian so $b_\pm^2 = 1 \pm a$. This is a consequence of functional calculus. Indeed, self-adjointness of multiplicative linear functionals and the fact that $r(a) = \|a\|$ implies that $\sigma(\pm a) \subseteq [-1, 1]$, and $1 \pm \mathrm{id}$ is positive in $\mathcal{C}[-1, 1]$, and hence admits a self-adjoint square root.

Now the Cauchy-Schwarz inequlity implies that $|f(a)| = |f(1^*a)| \leq f(1)f(a^*a)$. Hence it suffices to show that $|f(a)| \leq f(1)$ for any hermitian a with $\|a\| \leq 1$. As above write $1 \pm a = b_\pm^2 = b_\pm^* b_\pm$ so $0 \leq f(1 \pm a) = f(1) \pm f(a)$, so $f(a)$ is real and $|f(a)| \leq f(1)$. □

If f is a positive functional on a unital C*-algebra with $f(1) = 1$, we call f a *state*. The set of states on $\mathcal{A}$ will be denoted $\mathcal{S}(\mathcal{A})$. Notice that in the case that $\mathcal{A} \cong \mathcal{C}(\Omega)$, states are in bijective correspondence with regular Borel probability measures on Ω.

If $\mathcal{A}$ is unital and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation, i.e. a unital $*$-homomorphism, then for ξ in $\mathcal{H}$, $\pi_{\xi,\xi}(a) = \langle \pi(a)\xi|\xi\rangle$ is a positive functional, as can be easily checked. [In fact, if $\mathcal{A}$ is unital, we may use the above proposition, and the uniform boundedness principle to show that π is bounded, a weak version of our automatic continuity theorem, above.] $\pi_{\xi,\xi} \in \mathcal{S}(\mathcal{A})$ if and only if $\|\xi\| = 1$.

Below is one of the most beautiful constructions in all of operator algebras.

Gelfand-Naimark Construction. *Let $\mathcal{A}$ be a unital C*-algebra and f be a state. Then there is a Hilbert space $\mathcal{H}_f$, a representation $\pi_f : \mathcal{A} \to \mathcal{B}(\mathcal{H}_f)$ and a unit vector ξ_f such that $f = (\pi_f)_{\xi_f,\xi_f}$.*

Proof. We let $\mathcal{N}_f = \{n \in \mathcal{A} : f(n^*n) = 0\}$. The Cauchy-Schwarz inequlity implies that $\mathcal{N}_f = \{n \in \mathcal{A} : [n|a]_f = 0$ for all a in $\mathcal{A}\}$. Further, we have that $a\mathcal{N}_f \subseteq \mathcal{N}_f$ since $[an|b]_f = f(b^*an) = f((a^*b)^*n) = [n|a^*b]_f$. Hence $\mathcal{N}_f$ is a left ideal in $\mathcal{A}$, and we can define a map $\pi_0 : \mathcal{A} \to \mathcal{L}(\mathcal{A}/\mathcal{N}_f)$ (linear operators on $\mathcal{A}/\mathcal{N}_f$) by $\pi_0(b)(a + \mathcal{N}_f) = ba + \mathcal{N}_f$. Define $\langle a + \mathcal{N}_f | b + \mathcal{N}_f\rangle_f = [a|b]_f$, and check that this is well-defined. Then $\langle\cdot|\cdot\rangle_f$ defines an inner product on $\mathcal{A}/\mathcal{N}_f$, hence a norm. By Cauchy-Schwarz inequality

$$\|\pi_0(a)(b + \mathcal{N}_f)\|_f^2 = [ab|ab]_f = f(b^*a^*ab).$$

Observe that the functional $x \mapsto f(b^*xb)$ is positive on $\mathcal{A}$, and hence has norm $f(b^*1b) = f(b^*b) = \|b\|_f^2$. Hence we see $f(b^*a^*ab) \leq \|b\|_f^2 \|a^*a\|$, and it follows immediately that $\|\pi_0(a)(b + \mathcal{N}_f)\|_f^2 \leq \|a\|^2 \|b\|_f^2$. Hence $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{A}/\mathcal{N}_f)$ is bounded. Let $\mathcal{H}_f$ denote the completion of $\mathcal{A} \to \mathcal{A}/\mathcal{N}_f$, with the usual Hilbert space structure. Hence each $\pi_0(a)$ extends uniquely to an operator $\pi_f(a)$ on $\mathcal{H}_f$. Inspect that on the inner product space $\mathcal{A}/\mathcal{N}_f$ we have

$$\pi_0(ab) = \pi_0(a)\pi_0(b), \quad \pi_0(a^*) = \pi_0(a)^* \quad \text{and} \quad \pi_0(1) = I.$$

Hence, by density, $a \mapsto \pi_f(a)$ enjoys the same properties, and is thus a representation.

Let $\xi_f = 1 + \mathcal{N}_f$. Check that $\langle \pi_f(a)\xi_f | \xi_f \rangle_f = f(a)$. $\square$

We often call $(\pi_f, \mathcal{H}_f, \xi_f)$ a *Gelfand triple* associated to f. It can be shown that for another Gelfand triple $(\pi'_f, \mathcal{H}'_f, \xi'_f)$, there exists a unitary operator $u : \mathcal{H}_f \to \mathcal{H}'_f$ for which $u\pi_f(\cdot)u^* = \pi'_f$ (i.e. π_f and π'_f are unitarily equivalent). Hence, up to unitary equivalence, the Gelfand triple associated to f is unique. We observe that $\pi_f(\mathcal{A})\xi)f$ is dense in $\mathcal{H}_f$, hence we call ξ_f a *cyclic* vector for π_f. If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is any representation which admits a cyclic vector ξ, $\|\xi\| = 1$, then the Gelfand-Naimark representation generated by the state $\pi_{\xi,\xi}$ is unitarily equivalent to π.

The above construction is sometimes called the Gelfand-Naimark-Segal (GNS) construction. Segal's contribution is to show that pure states, i.e. extreme points of $\mathcal{S}(\mathcal{A})$ are in bijective correspondence with equivalence classes of irreducible representations. We will not require this level of refinement, but the interested reader should seek it out.

The Gelfand-Naimark construction suggests to us that in order to find "sufficiently many" representations of $\mathcal{A}$ on Hilbert space, we need "sufficiently many" states. Here "sufficiently many" will mean separating points on $\mathcal{A}$. Multiplicative linear functionals, hence probability measures, furnish sufficiently many states on commutaive C*-algebras. A combination of functional calculus and Hahn-Banch theorem will allow us to boost this up to an arbitrary C*-algebra.

We call elements of the form a^*a *positive*. Notice that if a is hermitian, then as observed in the proof of the proposition about boundedness of positive functionals, $\sigma(a) \subset \mathbb{R}$. By spectral mapping theorem $\sigma(a^2) = \{\lambda^2 : \lambda \in \sigma(a)\} \subset [0, \infty)$. We aim to show more generally that $\sigma(a^*a) \subset [0, \infty)$.

Lemma. **(i)** *If a, b are hermitian with $\sigma(a), \sigma(b) \subset [0, \infty)$, then $\sigma(a + b) \subset [0, \infty)$ too.*

(ii) *If $\sigma(a^*a) \subset (-\infty, 0]$, then $a = 0$.*

Proof. **(i)** We may rescale so $\|a\|, \|b\| \leq 1$. Since $\sigma(a) \subseteq [0, 1]$, $\sigma(1 - a) \subseteq [0, 1]$ by spectral mapping theorem. Hence since $1 - a$ is hermitian, $\|1 - a\| = r(1 - a) \leq 1$. Similarly $\|1 - b\| \leq 1$. Thus the hermitian element $1 - \frac{1}{2}(a + b) = \frac{1}{2}(1 - a) + \frac{1}{2}(1 - b)$ is of norm ≤ 1, and hence has spectrum within $[-1, 1]$. Applying spectral mapping, again, we find $\sigma(a + b) \subset [0, 4]$.

(ii) Since $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ (which is generally proved as an exercise in functional analysis), our assumption provides that $\sigma(aa^*) \subset (-\infty, 0]$. By part (i) we see that $\sigma(a^*a + aa^*) \subset (-\infty, 0]$. Let $x = \mathrm{Re}a$, $y = \mathrm{Im}a$ and note that $a^*a + aa^* = x^2 + y^2$. We have already commented that $\sigma(x^2), \sigma(y^2) \subseteq [0, \infty)$. Thus, employing (i), again, we see that $\sigma(x^2 + y^2) = \{0\}$. Thus $\|x^2 + y^2\| = r(x^2 + y^2) = 0$, so $x^2 = -y^2$. Again we see that $\sigma(x^2) = \{0\} = \sigma(y^2)$, and hence $\|x^2\| = 0 = \|y^2\|$. By functional calculus it follows that $x = y = 0$. □

Positive elements admit positive spectrum. *If a is an element of a unital C*-algebra $\mathcal{A}$, then $\sigma(a^*a) \subset [0, \infty)$.*

Proof. We know that $\sigma(a^*a) \subset \mathbb{R}$, since a^*a is hermitian. Define continuous functions on $\mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{t} & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} \sqrt{-t} & \text{if } t \leq 0 \\ 0 & \text{if } t > 0. \end{cases}$$

Notice that $f^2 - g^2 = \mathrm{id}$, $f^* = f$, $g^* = g$ and $fg = 0$, hence by functional calculus $u = f(a^*a)$ and $v = g(a^*a)$ satisfy $u^2 - v^2 = a^*a$, u, are hermitian, and $uv = 0$.

Now $va^*av = vu^2v - v^4 = -v^4$ and since $v^4 = (v^2)^2$ is hermitan, $\sigma((av)^*av) = \sigma(-v^4) \subset (-\infty, 0]$, so $v^4 = (av)^*av = 0$, and, as v is hermitian, we find $v = 0$ too. Hence $a^*a = u^2$ so $\sigma(a^*a) = \sigma(u^2) \subset [0, \infty)$. □

Observe a trivial consequence of the result above is that an element a of a unital C*-algebra is positive if and only if it is hermitian and $\sigma 9a) \subset [0, \infty)$.

We now gain a converse to the fact that positive functionals are bounded. We will require this to find sufficiently many states on $\mathcal{A}$.

Corollary. *If a linear functional f on a unital C*-algebra $\mathcal{A}$ satisfies $\|f\| = f(1) = 1$, then f is a state.*

Proof. Let us observe that, by functional calculus, a hermitian elements $a \neq 0$, b satisfy

$$a \text{ is positive} \quad \Leftrightarrow \quad \sigma\left(\frac{2}{\|a\|}a - 1\right) \subseteq [-1, 1]$$

$$\sigma(b) \subseteq [-1, 1] \quad \Leftrightarrow \quad \frac{1}{2}b + 1 \text{ is positive}$$

since the same properties hold for real numbers. Hence we need only to establish that for f, satisfying the assumptions above, that

$$f(b) \subseteq [-1,1] \text{ whenever } \sigma(b) \subseteq [-1,1].$$

To this end, we observe that $\bigcap_{t\in\mathbb{R}} D_{\sqrt{1+t^2}}(it) = [-1,1]$ (draw a picture). Hence functional calculus tells us that for hermitian b

$$\sigma(b) \subset [-1,1] \quad \Leftrightarrow \quad \|b+it\| \leq \sqrt{1+t^2} \text{ for all } t \text{ in } \mathbb{R}.$$

But then, for such b, we have

$$|f(b)+it| = |f(b+it1)| \leq \|b+it\| \leq \sqrt{1+t^2} \text{ for all } t \text{ in } \mathbb{R}$$

so $f(b) \subset [-1,1]$. □

Sufficiently Many States. *Let $\mathcal{A}$ be a unital C*-algebra. Then for any hermitian a in $\mathcal{A}$, there is f in $\mathcal{S}(\mathcal{A})$ for which $|f(a)| = \|a\|$.*

Proof. Consider first, the algebra $\mathrm{C}_1^*(a)$, from functional calculus. Let λ in $\sigma(a)$ be so $|\lambda| = r(a)$, and hence the multiplicative linear functional $\omega = \omega_\lambda$, which is a state (for b in $\mathrm{C}_1^*(a)$, $\omega(b^*b) = \overline{\omega(b)}\omega(b) \geq 0$, $\omega(1) = 1$) satisfies $|\omega(a)| = |\lambda| = \|a\|$. Let f be any Hahn-Banach extension of ω to all of $\mathcal{A}$. Then $\|f\| = 1 = f(1)$, so by the last corollary, f too is a state. □

We can now exploit all of the results of this section to show that C*-subalgebras of $\mathcal{B}(\mathcal{H})$ really are generic.

Gelfand-Naimark Theorem. *Given a C*-algebra $\mathcal{A}$, there is a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

Proof. First we suppose $\mathcal{A}$ is unital. Let D be a dense subset of $\mathcal{A}$. For each d in D let f_d in $\mathcal{S}(\mathcal{A})$ be so that $f_d(d^*d) = \|d\|^2$. Let $(\pi_d, \mathcal{H}_d, \xi_d)$ denote the Gelfand triple associated with f_d, so $\|\pi_d(d)\xi_d\|^2 = \langle \pi_d(d^*d)\xi_d|\xi_d\rangle_d = f_d(d^*d) = \|d\|^2$. Hence by contractivity of $*$-homomorphisms, we must have $\|\pi_d(d)\| = \|d\|$. Let $\mathcal{H} = \ell^2\text{-}\bigoplus_{d\in D}\mathcal{H}_d$ and $\pi = \bigoplus_{d\in D}\pi_d$.

If $\mathcal{A}$ is not unital, apply the above result to $\mathcal{A}_1$. The restrict the representation to $\mathcal{A}$. [See remark (†) below.] □

If $\mathcal{A}$ is separable, we can trace through the proof of the Gelfand-Naimark construction to see that $\mathcal{H}_f$ is always separable. Hence we can achieve a separable $\mathcal{H}$ in the theorem above.

(†) For unital $\mathcal{A}$, we insisted that representations satisfy $\pi(1) = 1$. For non-unital $\mathcal{A}$ we insist on the weaker condition that $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. It is not, at first blush, obvious how this is acheived. However, you are recommend to learn why C*-algebras always have *contractive approximate identities* (or, if one is a vulgarist, "bounded approximate identities of bound 1"). Some estimates are then required to show that $\pi_f(\mathcal{A})\mathcal{H}_f$ is dense in $\mathcal{H}_f$ for any state f in $\mathcal{S}(\mathcal{A}_1)$. Moreover, the image of the contractive approximate identity in $\mathcal{H}_f$ can be shown to admit a limit point, which will serve as the vector ξ_f. A remarkable factorisation theorem of P.J. Cohen then shows that $\pi_f(\mathcal{A})\mathcal{H}_f$ is actually closed.

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