von Neumann's Double Commutation Theorem

Gien a non-empty subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ we let its *commutatnt* be given by

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for each } S \text{ in } \mathcal{S}\}.$$

It is easy to verify that

- $\mathcal{S} \subseteq \mathcal{T}$ implies $\mathcal{S}' \supseteq \mathcal{T}'$;
- $\mathcal{S}'$ is always a WOT-closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing I;
- $\mathcal{S}'' = (\mathcal{S}')'$ contains $\mathcal{S}$, and $\mathcal{S}''' = \mathcal{S}'$.

Furthermore, if $\widetilde{\mathcal{S}} = \{S^* : S \in \mathcal{S}\} = \mathcal{S}$, then check too that $\mathcal{S}'$ is self-adjoint, hence a von Neumann algebra.

von Neumann's Double Commutation Theorem. *Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ be a non-degenerate C^*-subalgebra. Then*

$$\overline{\mathcal{A}}^{\mathrm{WOT}} = \mathcal{A}''$$

i.e. the weak operator topology closure is the same as the second commutant.

Proof. By comments above, its suffices to show that $\mathcal{A}'' \subseteq \overline{\mathcal{A}}^{\mathrm{WOT}}$. Since $\mathcal{A}$ is convex, we know that $\overline{\mathcal{A}}^{\mathrm{WOT}} = \overline{\mathcal{A}}^{\mathrm{SOT}}$. Thus if $T \in \mathcal{A}''$, and $x_1, \ldots, x_n$ in $\mathcal{H}$ are given, we wish to see for any $\epsilon > 0$ that there is A in $\mathcal{A}$ for which

$$\sum_{i=1}^{n} \|(A - T)x_i\|^2 < \epsilon^2 \tag{$\dagger$}$$

for then it follows that $\|(A - T)x_i\| < \epsilon$ for each i, i.e. the basic SOT neigbbourhood $\bigcap_{i=1}^{n} \{S \in \mathcal{B}(\mathcal{H}) : \|(S - T)x_i\| < \epsilon$ meets $\mathcal{A}$.

We identify $\mathcal{B}(\mathcal{H}^n) \cong \mathrm{M}_n(\mathcal{B}(\mathcal{H}))$ in the usual manner, i.e. as in the proof of the Kaplansky density theorem. Let

$$\mathcal{D}_{\mathcal{A}} = \left\{ \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A \end{bmatrix} : A \in \mathcal{A} \right\} \subset \mathrm{M}_n(\mathcal{B}(\mathcal{H})).$$

Then check that

$$\mathcal{D}'_{\mathcal{A}} = \mathrm{M}_n(\mathcal{A}') \quad \text{hence} \quad \mathcal{D}''_{\mathcal{A}} = \mathrm{M}_n(\mathcal{A}')' \supseteq \mathcal{D}_{\mathcal{A}''}.$$

The non-degeneracy of $\mathcal{A}$ entails that

$$\mathcal{M} = \overline{\mathcal{D}_{\mathcal{A}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} \text{ contains } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathcal{H}^n.$$

Let $P = P_{\mathcal{M}}$ denote the orthogonal projection onto $\mathcal{M}$ in $\mathcal{B}(\mathcal{H}^n) \cong \mathrm{M}_n(\mathcal{B}(\mathcal{H}))$. Since $\mathcal{M}$ is $\mathcal{D}_{\mathcal{A}}$-invariant, and since $\mathcal{D}_{\mathcal{A}}$ is a $*$-algebra (since $\mathcal{A}$ is a C*-subalgebra), we find that $\mathcal{M}$ is also reducing, i.e. $DP = PD$ for each D in $\mathcal{D}_{\mathcal{A}}$. Hence $XP = PX$ for each X in $\mathcal{D}''_{\mathcal{A}}$, in particular for any $X = \begin{bmatrix} T & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & T \end{bmatrix}$ in $\mathcal{D}_{\mathcal{A}''}$. In particular, $Xx \in \mathcal{M}$, and by definition of $\mathcal{M}$ there is $D = \begin{bmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A \end{bmatrix}$ in $\mathcal{D}_{\mathcal{A}}$ such that $\|Dx - Tx\|_2 < \epsilon$, which gives exactly (†). □

Relation to Borel functional calculus.

If N in $\mathcal{B}(\mathcal{H})$ is normal, then by Fuglede's Theorem (A3, Q3 (a)), $\{N\}' = \{N, N^*\}'$. Hence an application of the double commutatnt theorem gives

$$\{N\}'' = \overline{\mathrm{C}^*_e(N)}^{\mathrm{WOT}}.$$

In particular, since $\mathrm{C}^*_e(N)$ is commutative, so too must be $\{N\}''$.

Let us tie this with the Boral functional calculus of N.

Theorem. *If $\mathcal{H}$ is separable, and N in $\mathcal{B}(\mathcal{H})$ is normal, then there is a positive measure μ on $\sigma(N)$ for which the Borel functional calculus induces an isomorphism $L^\infty(\sigma(N), \mu) \cong \{N\}''$.*

Let me merely sketch a proof.

(I) *There is a measure for which*

$$\mathcal{N}_\mu = \{f \in L^\infty(\sigma(N), \mathcal{B}) : \mu(f^{-1}(\mathbb{C} \setminus \{0\})) = 0\}$$

is the kernel of the Borel functional calculus.

Since $\mathcal{H}$ is separable, C*-algebra $\mathrm{C}^*_e(N)$ admits a countable cyclic decomposition: there is an orthonormal set $\{x_1, x_2, \dots\}$ in $\mathcal{H}$ for which $\mathrm{C}^*_e(N)x_i \perp \mathrm{C}^*_e(N)x_j$, for $i \neq j$ and $\mathcal{H} = \ell^2\text{-}\bigoplus_{i=1,2,\dots} \overline{\mathrm{C}^*_e(N)x_i}$. Let $P_i = P_{\overline{\mathrm{C}^*_e(N)x_i}}$ denote the otrhogonal projection onto $\overline{\mathrm{C}^*_e(N)x_i}$. Then check that for x in $\mathcal{H}$ we have

$$\mu_{x,x} = \sum_{i=1}^{n} \mu_{P_i x, P_i x} \text{ where each } \mu_{P_i x, P_i x} \ll \mu_i := \mu_{x_i, x_i}.$$

[You will wish to have some comfort with measure theory to do this exercise.] But it then can be shown that

$$\text{each } \mu_{x,x} \ll \mu := \sum_{i=1,2,\dots} \frac{1}{2^i}\mu_i \text{ and } \bigcap_{x \in \mathcal{H}} \mathcal{N}_{\mu_{x,x}} = \mathcal{N}_\mu.$$

Notice that $L^\infty(\sigma(N), \mu) = L^\infty(\sigma(N), \mathcal{B})/\mathcal{N}_\mu$

(II) $\mathrm{b}(\mathcal{B}(\mathcal{H}))$ *is both WOT-metrizable and SOT-metrizable.*

Given a dense subset $\{z_i\}_{i=1}^n$ of $\mathrm{b}(\mathcal{H})$, consider the metrics

$$d_W(S,T) = \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} |\langle (S-T)x_i, x_j \rangle| \text{ and } d_S(S,T) = \sum_{i=1}^{\infty} \frac{1}{2^i} \|(S-T)x_i\|.$$

(III) *Finalé.*

Given $S \in \{N\}''$, Kapalansky's density theorem allows us to find a bounded net — which thanks to (II) we may choose to be a sequence — $(A_n)_{n=1}^{\infty}$ from $\mathrm{C}^*_e(N)$ for which $S = \text{WOT-}\lim_{n\to\infty} A_n$. Now we find f_n in $\mathcal{C}(\sigma(N))$ for which $A_n = f_n(N)$. We regard $\mathcal{C}(\sigma(N))$ a subspace of $L^\infty(\sigma(N), \mu) \cong L^1(\sigma(N), \mu)^*$. Since $\mathcal{C}(\sigma(N))$ is dense in $L^1(\sigma(N), \mu)$, the latter space is separable, and hence the unit ball, thus any ball, in $L^\infty(\sigma(N), \mu)$ is weak*-metrizable. Thus we may find a weak*-convergeng subsequence $(f_{n_k})_{k=1}^{\infty}$ of the bounded sequence of functions $(f_n)_{n=1}^{\infty}$, with weak*-limit f in $L^\infty(\sigma(N), \mu) = L^\infty(\sigma(N), \mathcal{B})/\mathcal{N}_\mu$. By abuse of notation, consider f as an element of $L^\infty(\sigma(N), \mathcal{B})$, i.e. we really mean $f = f + \mathcal{N}_\mu$, and consider $f(N)$. Then for x, y in $\mathcal{H}$, we have $\mu_{x,y} \ll \mu$, and Radon-Nikodym provides $g_{x,y}$ in $L^1(\sigma(N), \mu)$ so $d\mu_{x,y} = g_{x,y} d\mu$. Hence

$$\langle f(N)x, y\rangle = \int_{\sigma(N)} f g_{x,y}\, d\mu = \lim_{k\to\infty} \int_{\sigma(N)} f_{n_k} g_{x,y}\, d\mu = \lim_{k\to\infty} \langle f_{n_k}(N)x, y\rangle$$

so $f(N) = \text{WOT-}\lim_{k\to\infty} f_{n_k}(N) = S$.

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