On the Character space of $L^\infty[0,1]$.

Theorem. *There are $2^{\mathfrak{c}}$ elements γ of $\Gamma_{L^\infty[0,1]}$ for which $\gamma|_{\mathcal{C}[0,1]} = \delta_{1/2}$.*

Let us begin with the following, whose obvious proof we omit.

Proposition. *The subalgebra*

$$\mathcal{L} = \left\{ f \in L^\infty[0,1] : 1_{[1/(n+1),1/n]} f = \alpha_n 1_{[1/(n+1),1/n]} \text{ for all } n \right\}$$

is isometrically isomorphic to $\ell^\infty(\mathbb{N})$.

Proposition. (Schur's Lemma) *For a commutative C^*-algebra $\mathcal{A}$, the Gelfand spectrum $\Gamma_{\mathcal{A}}$ is the space of pure states.*

Proof. A $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is irreducible if and only if its commutant $\pi(\mathcal{A})'$ — which is a von Neuman algebra — contains no non-trivial projections. If $S \in \pi(\mathcal{A})'$, then $\mathrm{Re}S, \mathrm{Im}S \in \pi(\mathcal{A})'$, and Borel functional calculus provides non-trivial projections in $\pi(\mathcal{A})'$ unless both $\mathrm{Re}S$ and $\mathrm{Im}S$ are scalar multiples of I. Hence, if π is irreducible, then $\pi(\mathcal{A})'' = (\mathbb{C}I)' = \mathcal{B}(\mathcal{H})$. Thus if $\mathcal{A}$ is commutative, so too must be $\pi(\mathcal{A})'' = \overline{\pi(\mathcal{A})}^{\mathrm{WOT}}$ (von Neumann's double commutation theorem), which entails that $\dim \mathcal{H} = 1$, so $\pi \in \Gamma_{\mathcal{A}}$ and is, itself, a state. $\square$

Below, we shall liberally use the rules of cardinal arithmetic, including the identity $2^{\aleph_0} = \mathfrak{c}$.

Thoerem. *$\beta\mathbb{N} = \Gamma_{\ell^\infty(\mathbb{N})}$ has cardinality $2^{\mathfrak{c}}$.*

Proof. We first note that $\mathfrak{c} \le |\ell^\infty(\mathbb{N})| \le |\mathbb{C}^{\mathbb{N}}| = \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = \mathfrak{c}$, and hence

$$|\beta\mathbb{N}| \le |\ell^\infty(\mathbb{N})^*| \le |\mathbb{C}^{\ell^\infty(\mathbb{N})}| = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}.$$

It remains to prove the non-trivial inequality of cardinals.

Let $I = [0,1]$, we show that the product space I^I is separable. Then, by the universal property of $\beta\mathbb{N}$, we find that $|\beta\mathbb{N}| \ge |I^I| = \mathfrak{c}^{\mathfrak{c}} = (2^{\aleph_0})^{\mathfrak{c}} = 2^{\mathfrak{c}}$. First, enumerate $\mathbb{Q} \cap [0,1] = \{r_n\}_{n=1}^\infty$. For each finite subset $\{J_1, \ldots, J_m\}$ of subintervals of $[0,1]$ with rational endpoints, and each finite set $\{n_1 < \cdots < n_m\}$ of $\mathbb{N}$ let

$$p_{J_1,\ldots,J_n,n_1,\ldots,n_m}(x) = \begin{cases} r_{n_i} & \text{if } x \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

The collection of such elements $p_{q_1,\ldots,q_m,n_1,\ldots,n_m}$ in I^I, indexed over all such finite subsets is countable (exercise). Furthermore, if $p_0 \in I^I$, a basic open neighbourhood of p_0 is of the form

$$U = \bigcap_{i=1}^{m} \left\{ p \in I^I : p_0(x_k) - \epsilon_k < p(x_k) < p_0(x_k) + \epsilon_k \right\} \quad \begin{array}{c} \text{for some } x_1, \dots, x_m \text{ in } I \\ \text{and } \epsilon_1, \dots \epsilon_n > 0. \end{array}$$

Notice that we may find $(r_{n_1}, \dots, r_{n_m})$ in $\prod_{k=1}^m (p_0(x_k) - \epsilon_k, p_0(x_k) + \epsilon_k)$, with $n_1 < \cdots < n_m$, and hence intervals $J_1, \dots, J_m$ with rational endpoints so $x_k \in J_k$. Hence $U \ni p_{J_1,\dots,J_m,n_1,\dots,n_m}$. Hence the collection of such points is dense in I^I. $\square$

Corollary. $|\Gamma_{L^\infty[0,1]}| = 2^{\mathfrak{c}}$.

Proof. Letting $\mathcal{L}$ be as in the first proposition, above, we have every element of $\Gamma_{\mathcal{L}} \cong \beta\mathbb{N}$ is a state, and hence extends to a pure state on $L^\infty[0,1]$, which by the secong proposition is a character. Thus $|\Gamma_{L^\infty[0,1]}| \geq \left| \Gamma_{L^\infty[0,1]}|_{\mathcal{L}} \right| = |\Gamma_{\mathcal{L}}| = 2^{\mathfrak{c}}$.

Conversely, since $L^\infty[0,1]$ is a quotient space of the Borel measurable function space $L^\infty([0,1],\mathcal{B})$, we have $|\Gamma_{L^\infty[0,1]}| \leq |\Gamma_{L^\infty([0,1],\mathcal{B})}| \leq |\Gamma^*_{L^\infty([0,1],\mathcal{B})}| \leq |\mathbb{C}^{[0,1]}| = 2^{\mathfrak{c}}$. $\square$

Proof of main result. For any s in $[0,1]$, let $G_s = \{\gamma \in \Gamma_{L^\infty[0,1]} : \gamma|_{\mathcal{C}[0,1]} = \delta_s\}$.

Given any two points $s < t$ in $(0,1)$, there is an order-preserving homeomorphism α on $[0,1]$ for which $\alpha(s) = t$, and α takes Lebesgue null sets to Lebesgue null sets; say, take $\alpha(x)$ to be $\frac{t}{s}x$ for $0 \leq x \leq s$ and $\frac{1-t}{1-s}(x-s)+t$ for $s < x \leq 1$. Then $f \mapsto A_\alpha(f) = f \circ \alpha$ is an automorphism of $\mathcal{C}[0,1]$, and also of $L^\infty[0,1]$ (composition of Borel functions is Borel). Thus $A^*_\alpha(\Gamma_{L^\infty[0,1]}) = \Gamma_{L^\infty[0,1]}$ and induces a homeomprhism on that space. Hence for $|G_t| = |A^*_\alpha(G_s)| = |G_s|$.

We wish to see too that $|G_0|, |G_1| \leq |G_{1/2}|$. Let $\alpha : [0,1] \to [0,1]$ be given by $\alpha(t) = 1/2 - t$ for $0 \leq t \leq 1/2$ and $\alpha(t) = 3/2 - t$ for $1/2 < t \leq 1$. But then A_α, defined similarly as above, is an automprphism of $L^\infty[0,1]$ and $A^*_\alpha(G_0 \cup G_1) \subseteq G_{1/2}$.

Combining the observations above, we use an infinite "pigeonhole principle", attempting to uniformly distribute $2^{\mathfrak{c}} = |\Gamma_{L^\infty[0,1]}|$ elements into $\mathfrak{c} = |[0,1]|$ sets $\{G_s\}_{s\in[0,1]}$, to see that $G_{1/2} = 2^{\mathfrak{c}}$. $\square$

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