# PMATH 810, Winter 2015

**Assignment #4** Due: “Friday” April 6.

Notational convention. If $T : \mathcal{H} \to \mathcal{L}$ is a bounded linear map between Hilbert spaces, write $T^* : \mathcal{L} \to \mathcal{H}$ be so $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for $x$ in $\mathcal{H}$ and $y$ in $\mathcal{L}$. If $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{L}}$ we call $U$ a *unitary*. Notice that unitaries are precisely the invertible isometries between Hilbert spaces. The proof is an easy adaptation of the characterization of partial isometries, given in class.

1. Let $\mathcal{A}$ be a unital C*-algebra and $0 \le a \le b$ in $\mathcal{A}$.

   **(a)** Show that if $a, b \in \mathrm{GL}(\mathcal{A})$, then $0 \le b^{-1} \le a^{-1}$.

   **(b)** Show that $0 \le a^{1/2} \le b^{1/2}$. [Hint. Show that $\|a^{1/2}(b+\varepsilon e)^{-1/2}\| \le 1$ for any $\varepsilon > 0$, the consider $r((b+\varepsilon e)^{-1/4}a^{1/2}(b+\varepsilon e)^{-1/4})$.]

   **(c)** Show that $0 \le a^2 \le b^2$ if $ab = ba$, but that this inequality fails, generally. [Try to obtain the failure in $\mathrm{M}_2(\mathbb{C}) = \mathcal{B}(\ell^2(2))$.]

2. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$, the compact operators on an infinite-dimensional separable Hilbert space. For any set $I$ let $\mathcal{H}^{(I)} = \{(x_i)_{i\in I} : \text{each } x_i \in \mathcal{H} \text{ and } \sum_{i\in I} \|x_i\|^2 < \infty\}$ denote the $I$-*amplification* of $\mathcal{H}$.

   **(a)** (Representations of $\mathcal{K}$) Show that for any non-zero $*$-representation $\pi : \mathcal{K} \to \mathcal{B}(\mathcal{L})$ ($\mathcal{L}$ another Hilbert space) which is *non-degenerate*, $\overline{\mathrm{span}\pi(\mathcal{K})\mathcal{L}} = \mathcal{L}$, there is a set $I$ and a unitary $U : \mathcal{L} \to \mathcal{H}^{(I)}$ such that $U\pi(K)U^*(x_\iota)_{i\in I} = (Kx_\iota)_{\iota\in I}$.

   [Hint. Fix an o.n.b. $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$, and let $E_{ij} = \langle\cdot, e_j\rangle e_i \in \mathcal{K}$ — the set of these is called a “matrix unit”. Notice that $E_{ij}^* = E_{ji}$ and $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Show that the subspaces $\pi(E_{jj})\mathcal{L}$ are mutually orthogonal, and isomorphic, thus each has o.n.b. of the same size: $(f_{j\iota})_{\iota\in I}$.]

   **(b)** Show that $\mathcal{K}$ is *simple*: the only non-zero closed ideal of $\mathcal{K}$ is itself.

   **(c)** Show that the only norm-closed ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}, \mathcal{K}$ and $\mathcal{B}(\mathcal{K})$. [Hint. One needs only to study “principal ideals” $\overline{\mathcal{B}(\mathcal{H})S\mathcal{B}(\mathcal{H})}$.]

   **(c’)** Does (c) hold without the assumption of separability of $\mathcal{H}$?

   **(d)** (Representations of the Calkin algebra) Show that if $\rho : \mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K} \to \mathcal{B}(\mathcal{L})$ is a non-zero $*$-representation, then $\mathcal{L}$ cannot be separable. [Hint. There exists an uncountable family $\mathcal{F}$ of subsets of $\mathbb{N}$ for which two distinct elements admit finite intersection (see my solution for A1 Q4 (c). Make a useful family $\{P_F\}_{\mathcal{F}}$ of projections on $\mathcal{H}$. Consider what happens to $\{\rho(P_F + \mathcal{K})\}_{F\in\mathcal{F}}$.]

[Remark. We deduce that any non-zero $*$-homomorphism $\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with $\rho(I) = I$ must be of a form suggested in (a), above. Indeed, (a) gives the structure of $\pi = \rho|_{\mathcal{K}}$, and we see that $\rho(S)\pi(K) = \rho(SK)$.]

3. **(a)** (Uniqueness of GNS construction) Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of a C*-algebra having cyclic vector $x$, $\|x\| = 1$. Let $\omega(a) = \langle \pi(a)x, x\rangle$ for $a$ in $\mathcal{A}$. Show that there is a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ which satisfies $U\pi_\omega(a) = \pi(a)U$ and $Ux_\omega = x$.

   **(b)** On $L^\infty[0,1]$, let $\omega(f) = \int_0^1 f\,dm$ (Lebesgue measure), and $\mu_m : L^\infty[0,1] \to \mathcal{B}(L^2[0,1])$ be given by $\mu_m(f)h(s) = f(s)h(s)$ for a.e. $s$ in $[0,1]$, and $h$ in $L^2[0,1]$. Show that there is a unitary $U : \mathcal{H}_\omega \to L^2[0,1]$ which *interwines* $\pi_\omega$ and $\mu_m$, i.e. $U\pi_\omega(f) = \mu_m(f)U$.

   **(c)** Show that each $\mu_m$-invariant subspace $\mathcal{M}$ of $L^2[0,1]$ is cyclic, and furthermore that if $\mathcal{M} \neq \{0\}$, then $\mathcal{M}$ admits a $\mu_m$-invariant subspace $\mathcal{N}$, with $\{0\} \subsetneq \mathcal{N} \subsetneq \mathcal{M}$. [Hint. Show that if $P\mu_m(f) = \mu_m(f)P$ for all $f$, then $P1 \in L^\infty[0,1]$ and $P = \mu_m(P1)$.]

   Hence we say that "$\pi_\omega$ admits no irreducible subrepresentation".

4. (Some applications of Borel functional calculus)

   **(a)** Show that for any unitary $U$ on a Hilbert space $\mathcal{H}$ that there is a Hermitian operator $H$ for which $U = \exp(iH)$.

   **(b)** Deduce that $\mathrm{GL}(\mathcal{H})$ is connected.

   **(c)** Let $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ be a von-Neumann algebra. Show that the extreme points of $\mathrm{b}(\mathcal{W}_h)$ consists of *symmetries*: $S$ in $\mathcal{N}$ such that $S^* = S$ and $S^2 = 1$.

   **(d)** Show that if $\mathcal{H}$ is separable, then each extreme point of $\mathrm{b}(\mathcal{B}(\mathcal{H}))$ is a partial isometry $U$ which is injective and/or surjective.

   [Hint. A partial isometry times a unitary remains a partial isometry. You may wish to use either (c), or that for $t \in \mathbb{R}$ with $|t| \leq 1$ then $t = \frac{1}{2}[(t + i\sqrt{1-t^2}) + (t - i\sqrt{1-t^2})]$.]