PMATH 810, Winter 2015

Assignment #3 Due: March 19.

We always let m denote the Lebesgue measure on $\mathbb{R}$, below.

1. **(a)** Show that $L^p(\mathbb{R})$ $(1 \leq p < \infty)$ admits the approximation property. [Hint. Devise a partial order on the set N of finite collections of measurable sets $\nu = \{E_1, \dots, E_n\}$ such that each E_i satisfies $0 < m(E_i) < \infty$ and $E_i \cap E_j = \varnothing$ for $i \neq j$, in such a manner that the operators $P_\nu f = \sum_{i=1}^n \frac{\int_{E_i} f\, dm}{m(E_i)} 1_{E_i}$ seem useful.]

   (b) Let K be a compact Hausdorff space, and $\{U_1, \dots, U_n\}$ be a finite open cover. Show that $\mathcal{C}(K)$ admits a *partition of unity, subordinate to* $\{U_1, \dots, U_n\}$: a family $\{h_1, \dots, h_n\}$ in $\mathcal{C}(K)$ such that

   $$h_j(K) \subset [0,1],\ \mathrm{supp} h_j \subset U_j, \text{ for } j = 1, \dots, n, \text{ and } h_1 + \cdots + h_n = 1.$$

   (c) Show that for a compact Hausdorff space K, $\mathcal{C}(K)$ has approximation property.

2. (Integral operators) Let $k \in \mathcal{C}([0,1] \times [0,1])$. Define for f in $L^p[0,1]$ $(1 \leq p < \infty)$

   $$Kf(x) = \int_0^1 k(x,y) f(y)\, dm(y) \text{ for a.e. } y \text{ in } [0,1].$$

   (a) Show that K takes $L^p[0,1]$ into itself, and, further, $K \in \mathcal{K}(L^p[0,1])$.

   (b) Let $k(x,y) = \max\{0, x-y\}$. Show that $\sigma_p(K) = \varnothing$, hence $\sigma(K) = \{0\}$. [Hint. Show that, in fact $\mathrm{ran} K \subseteq \mathcal{C}[0,1]$, and $\mathrm{ran} K^2 \subseteq \mathcal{C}^1[0,1]$.]

3. Let N, M in $\mathcal{B}(\mathcal{H})$ each be normal: $N^*N = NN^*$ and $M^*M = MM^*$.

   (a) Suppose S in $\mathcal{B}(\mathcal{H})$ satisfies $MS = SN$. Show that $M^*S = SN^*$. [Hint. If $A^* = -A$ then $\exp A$ is unitary. Show that $\exp(M^*)S \exp(-N^*) = \exp(M^* - M)S \exp(N - N^*)$ to deduce that for f in $\mathcal{B}(\mathcal{H})^*$, $z \mapsto f(\exp(zM^*)S \exp(-zN^*))$ on $\mathbb{C}$ is bounded.]

   (b) Show that if M and N are *similar*, i.e. there is $S \in \mathrm{GL}(\mathcal{H})$ for which $SNS^{-1} = M$, then M and N are *unitarily equivalent*, i.e. there is U for which $U^*U = I = UU^*$ and $UNU^* = M$.

4. **(a)** Let X be a topological space, and

$$\mathcal{C}_\beta(X) = \left\{ f : X \to \mathbb{C} \,\middle|\, f \text{ is continuous and } \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty \right\}.$$

[You may take for granted that $\mathcal{C}_\beta(X)$ is complete with respect to the norm $\|\cdot\|_\infty$. Also, under pointwise operations, $\mathcal{C}_\beta(X)$ is a C*-algebra.] Moreover, show that its spectrum $\beta X = \Gamma_{\mathcal{C}_\beta(X)}$ satisfies that $x \mapsto \delta_x$ is continuous and $\{\delta_x : x \in X\}$ is dense in βX. Moreover, βX satisfies the following universal property: for any compact Hausdorff space K and continuous map $\eta : X \to K$, there is a unique continuous map $\beta\eta : \beta X \to K$ such that $\beta\eta(\delta_x) = \eta(x)$, i.e. the following diagram commutes.

$$\begin{tikzcd} \beta X \arrow[r, "\beta\eta"] & K \\ X \arrow[u, "x \mapsto \delta_x"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

[Culture. We say that X is *regular* if $\mathcal{C}_\beta(X)$ separates points of X; this is equivalent to saying $x \mapsto \delta_x : X \to \beta X$ is injective.]

(b) Assume that X is Hausdorff. Show that X is locally compact if and only if $\{\delta_x : x \in X\}$ is homeomorphic to X and open in βX.

(c) Let $\mathcal{L}$ the σ-algebra of Lebesgue-measurable subsets of $[0,1]$. A *Lebesgue pseudo-ultrafilter* is any family $\mathcal{U} \subset \mathcal{L}$ consisting of non-null sets, such that for E, F in $\mathcal{U}$, G in $\mathcal{L}$ we have

$$E \cap F \in \mathcal{U}; \text{ if } E \subseteq G \text{ then } G \in \mathcal{U}; \text{ and either } G \in \mathcal{U} \text{ or } [0,1] \setminus G \in \mathcal{U}.$$

Show that the span of idempotents, $\mathcal{S} = \operatorname{span}\{1_E : E \in \mathcal{L}\}$ is dense in $L^\infty[0,1]$, and that the linear functional on $\mathcal{S}$ determined by $\delta_{\mathcal{U}}(1_E) = 1$ if $E \in \mathcal{U}$ and $\delta_{\mathcal{U}}(1_E) = 0$ if $E \notin \mathcal{U}$, uniquely extends to an element of $\Gamma_{L^\infty[0,1]}$, and all elements of $\Gamma_{L^\infty[0,1]}$ are given this way.

(d) $\mathcal{C}[0,1]$ identifies naturally as a closed subspace of $L^\infty[0,1]$. Show that there are infinitely many Lebesgue pseudo-ultrafilters $\mathcal{U}$ for which $\delta_{\mathcal{U}}(f) = f(\frac{1}{2})$ for f in $\mathcal{C}[0,1]$.

5. Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{F}_N(\mathcal{H})$ denote the set of bounded finite rank nilpotent operators on $\mathcal{H}$, i.e. those F in $\mathcal{F}(\mathcal{H})$ such that $F^n = 0$ for some n in $\mathbb{N}$. Compute the closure $\overline{\mathcal{F}_N(\mathcal{H})}$ in $\mathcal{B}(\mathcal{H})$. [Hint. You may find A2, Q2 handy. Also look up "Schur's Theorem" in linear algebra.]