PMATH 810, Winter 2015

Assignment #2 Due: February 26.

1. (Kakutani's shift operator.) Let for k, n in $\mathbb{N}$

$$s_k = \frac{1}{\gcd(k, 2^k)} \quad \text{and} \quad s_{n,k} = \begin{cases} s_k & \text{if } s_k > \frac{1}{2^n} \\ 0 & \text{if } s_k \leq \frac{1}{2^n}. \end{cases}$$

Let $1 \leq p < \infty$, and $S, S_n \in \mathcal{B}(\ell^p)$ be given on a dense spanning set by

$$Se_k = s_k e_{k+1} \text{ and } S_n e_k = s_{k,n} e_{k+1}$$

where $e_k = (0, \dots, 0, 1, 0, \dots)$, i.e. 1 in the kth position, only.

(a) Compute the spectral radii $r(S)$ and $r(S_n)$, and also show that $\lim_{n\to\infty} \|S - S_n\| = 0$. [Hint. First compute $\|S^{2^n}\|$.]

Remark. Hence we conclude that $a \mapsto r(a)$ is not continuous on a general Banach algebra, and, furthermore, lower semi-continuity of the spectrum (defined in (b), below) fails in a general Banach algebra.

(b) Show that if a Banach algebra $\mathcal{A}$ is commutative, then $a \mapsto r(a)$ is continuous, and we have lower semi-continuity of the spectrum at a in $\mathcal{A}$: given $\varepsilon > 0$, there is $\delta > 0$ such that for any b in $\mathcal{A}$ we have

$$\|a - b\| < \delta \quad \Rightarrow \quad \sigma(a) \subset \sigma(b)_\varepsilon := \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(b)) < \varepsilon\}.$$

2. Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}$.

(a) (Upper semicontinuity of the spectrum at a.) Given $\varepsilon > 0$, show that there is $\delta > 0$ such that for any b in $\mathcal{A}$ we have

$$\|a - b\| < \delta \quad \Rightarrow \quad \sigma(b) \subset \sigma(a)_\varepsilon$$

[Hint: Show that $z \mapsto \|(ze - a)^{-1}\|$ on $\mathbb{C} \setminus \sigma(a)_\varepsilon$ is maximized on $\partial[\sigma(a)_\varepsilon]$.]

(b) Show that if $\sigma(a)$ is not connected, then there is $\delta > 0$ so for b in $\mathcal{A}$ with $\|a - b\| < \delta$, $\sigma(b)$ is not connected. [Use δ from (a), let $b_t = (1-t)a - tb$, and consider suitable associated Riesz projections e_t. Show that $t \mapsto e_t$ is continuous.]

3. Let $\mathcal{A}$ be a unital Banach algebra with identity e.

   (a) Show that if $ab = ba$ in $\mathcal{A}$ then $\exp(a+b) = \exp(a)\exp(b)$,

   (b) Show, by way of an example, that the result of (a) may fail if $ab \neq ba$.

   (c) Show that if $\|e - a\| < 1$, then there is b in $\mathcal{A}$ for which $a = \exp(b)$.

   (d) Let $\mathrm{GL}(\mathcal{A})_e = \{\exp(a_1)\ldots\exp(a_n) : a_1, \ldots, a_n \in \mathcal{A}, n \in \mathbb{N}\}$. Show that $\mathrm{GL}(\mathcal{A})_e$ is an open, normal subgroup of $\mathrm{GL}(\mathcal{A})$. Moreover, show that $\mathrm{GL}(\mathcal{A})_e$ is connected, and hence the connected component of the identity in $\mathrm{GL}(\mathcal{A})$.

   (e) Compute $\mathrm{GL}(\mathcal{C}(\mathbb{T}))/\mathrm{GL}(\mathcal{C}(\mathbb{T}))_e$. [You may need a "useful" description of $\mathrm{GL}(\mathcal{C}(\mathbb{T}))_e$, first.]

4. Let M_n denote $n \times n$ $\mathbb{C}$-matrices, $a \in \mathrm{M}_n$.

   (a) Compute all of the Riesz idempotents associated with a.

   (b) When is each such idempotent in $\langle a \rangle_e$? In $\langle a \rangle$?

   (c) When is a in the linear span of its Riesz idempotents?

5. Let $\mathcal{A}$ be a commutative Banach algebra.

   (a) Suppose that $\mathcal{A}$ is semi-simple, $\Gamma_{\mathcal{A}}$ is compact and there is a u in $\mathcal{A}$ for which $0 \notin \hat{u}(\Gamma_{\mathcal{A}})$. Show that $\mathcal{A}$ is unital. [Hint. Work in the unitization $\widetilde{\mathcal{A}}$ with identity $\tilde{e} = (0, 1)$. The identity $u(z\tilde{e} - u)^{-1} = \frac{1}{z}\tilde{e} - (z\tilde{e} - u)^{-1}$ can be used to show that a certain Riesz idempotent e for u is in $\mathcal{A}$, and serves as its identity.]

   (b) Suppose that $\widehat{\mathcal{A}} \subseteq \mathcal{C}(\Gamma_{\mathcal{A}})$ is *conjugate closed*: for a in $\mathcal{A}$, there is a^* in $\mathcal{A}$ for which $\widehat{a^*} = \overline{\hat{a}}$. Show that if $\mathcal{A}$ is regular, then it is also *normal*: given closed $F \subset \Gamma_{\mathcal{A}}$ and compact $L \subset \Gamma_{\mathcal{A}}$, there is e_L in $\mathcal{A}$ for which $\widehat{e_L}|_L = 1$ and $\widehat{e_L}|_F = 0$. [Let $\mathcal{A}_L = \mathcal{A}/\mathrm{k}(L)$. Show that $\mathcal{A}_L$ is semisimple with spectum $\cong L$, and satisfies assumptions of (a).]

   (c) Show that if $\mathrm{span}\{u \in \mathcal{A} : u^2 = u\}$ (span of idempotents) is dense in $\mathcal{A}$, then $\Gamma_{\mathcal{A}}$ is 0-dimensional (in the usual w*-topology), i.e. for any $\gamma \neq \chi$ in $\Gamma_{\mathcal{A}}$, there are open $U \ni \gamma$ and $V \ni \chi$ so $\Gamma_{\mathcal{A}} = U \cup V$ while $U \cap V = \varnothing$.

   (d) Show that if $\mathcal{A}$ is unital, regular and semisimple, and $\widehat{\mathcal{A}}$ is conjugate closed and $\Gamma_{\mathcal{A}}$ is 0-dimensional, then $\mathcal{B} = \overline{\mathrm{span}\{u \in \mathcal{A} : u^2 = u\}}$ admits $\Gamma_{\mathcal{B}} = \Gamma_{\mathcal{A}}$.