PMATH 810, Winter 2015

Assignment #1 Due: January 26.

1. **(a)** Show that if S and T are semigroups, then there is an isometric isomorphism $\Phi : \ell^1(S) \to \ell^1(T)$ if and only if S and T are isomorphic semigroups.

   (b) Show, by way of an example, that the above result fails if we do not assume that Φ is isometric.

2. Let S in $\mathcal{B}(\ell^p)$ $(1 < p < \infty)$ be the unilateral shift, given by $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Compute $\sigma_{\mathrm{ap}}(S)$ and $\sigma_{\mathrm{res}}(S)$.

3. A *weight* on a commutative semi-group is a function $\omega : S \to (0, \infty)$ for which $\omega(s+t) \leq \omega(s)\omega(t)$ (we write $(s,t) \mapsto s+t$ for the semigroup product).

   (a) Let $\ell^1(S, \omega) = \left\{ f : S \to \mathbb{C} \,\middle|\, \|f\|_{\omega,1} = \sum_{s \in S} |f(s)|\omega(s) < \infty \right\}$. Show that under usual convolution, $\ell^1(S, \omega)$ is a Banach algebra. Moroever, if two weights ω and ω' are *equivalent* – there exists $c, C > 0$ for which $c\omega \leq \omega' \leq C\omega$ – then there is an isomorphism $\ell^1(S, \omega) \cong \ell^1(S, \omega')$.

   (b) Show that there are natural homeomorphisms

   $$\Gamma_{\ell^1(\mathbb{N}_0, \omega)} \cong \omega_+ \overline{\mathbb{D}} \quad \text{and} \quad \Gamma_{\ell^1(\mathbb{Z}, \omega)} \cong \mathbb{A}(0, \tfrac{1}{\omega_-}, \omega_+)$$

   where $\omega_\pm = \lim_{n\to\infty} \omega(\pm n)^{1/n}$ (show these limits exist), and $\mathbb{A}(0, r, R) = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ for $0 < r \leq R < \infty$.

   Examples. Weights with nth value $\omega(n)$, include $(1 + |n|)^\alpha$ $(\alpha > 0)$, $e^{\alpha|n|}$ and also $\|T^n\|$ (T a bounded operator on a Banach space; with T invertible in the case we want the weight definable on $\mathbb{Z}$).

   (c) Let $\mathcal{X}$ be a Banach space and $T \in \mathrm{GL}(\mathcal{X})$, with $\sigma(T) \subset \mathbb{A}(0, \frac{1}{r}, r)$ $(r \geq 1)$. Given weight defined by $\omega(n) = (r + \varepsilon)^{|n|}$ $(\varepsilon > 0)$, show that

   $$f \mapsto \sum_{n \in \mathbb{Z}} f(n) T^n$$

   is a bounded homomorphism from $\ell^1(\mathbb{Z}, \omega)$ into $\mathcal{B}(\mathcal{X})$.

4. **(a)** Let I be an infinite set with power set $\mathcal{P}(I)$. Let

$$\mathrm{m}(I) = \left\{ \mu : \mathcal{P}(I) \to \mathbb{C} : \begin{array}{c} \mu(E \cup F) = \mu(E) + \mu(F) \text{ whenever} \\ E \cap F = \varnothing \text{ and } V(\mu) < \infty \end{array} \right\}$$

$$\text{where } V(\mu) = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| : \begin{array}{c} E_1, \ldots, E_n \in \mathcal{P}(I) \text{ with} \\ E_i \cap E_j = \varnothing \text{ if } i \neq j,\ (n \in \mathbb{N}) \end{array} \right\}.$$

Show that for each φ in the dual space $\ell^\infty(I)^*$, there is an unique μ in $\mathrm{m}(I)$ such that $\varphi(1_E) = \mu(E)$ for every E in $\mathcal{P}(I)$. Moroever, any μ in $\mathrm{m}(I)$ is obtained by some $\varphi = \varphi_\mu$ in $\ell^\infty(I)$ as above, and $\|\varphi_\mu\| = V(\mu)$.

[Consider the dense subspace of simple functions in ℓ^∞.]

Hereafter, we shall write simply $\mu(f)$, instead of $\varphi_\mu(f)$.

(b) Show that $\varphi_\mu \in \Gamma_{\ell^\infty(I)}$ if and only if $\mu(E \cap F) = \mu(E)\mu(F)$ and hence $\mu(E) \in \{0, 1\}$. Furthermore, show that $\mathcal{U}_\mu = \{E \in \mathcal{P}(I) : \mu(E) = 1\}$ is a *filter* on I:

$$\varnothing \notin \mathcal{U}_\mu, \quad E \in \mathcal{U}_\mu, E \subset F \ \Rightarrow\ F \in \mathcal{U}_\mu, \quad \text{and } E, F \in \mathcal{U}_\mu \ \Rightarrow\ E \cap F \in \mathcal{U}_\mu$$

and, further, satisfies the condition that for any $E \in \mathcal{P}(I)$, exactly one of E or $I \setminus E$ is in $\mathcal{U}_\mu$. We call a filter satisfying the last condition an *ultrafilter*.

(c) Let $\mathcal{F} = \{I \setminus F : F \text{ in } \mathcal{P}(I) \text{ is finite}\}$ be the co-finite filter. Show that there are at least $|I||\mathbb{R}|$ ultrafllters containing $\mathcal{F}$.

Remark. Any ultrafilter $\mathcal{U}$ gives an element $\delta_\mathcal{U}$ of $\mathrm{m}(I)$ given by $\delta_\mathcal{U}(E) = \begin{cases} 1 & \text{if } E \in \mathcal{U} \\ 0 & \text{if } E \notin \mathcal{U} \end{cases}$. If $i \in I$, the *trivial* ultrafilter at i is given by $\mathcal{U}_i = \{E \in \mathcal{P}(I) : i \in E\}$. Write $\delta_i = \delta_{\mathcal{U}_i}$.

(d) Write $\beta I = \Gamma_{\ell^\infty(I)} = \{\delta_\mathcal{U} : \mathcal{U} \text{ is an ultrafilter on } I\}$, which is a compact space with weak*-topology. Show that the family $\{\delta_i\}_{i \in I}$ is dense in $\Gamma_{\ell^\infty(I)}$. Furthermore, if K is any compact Hausdorff space and $\eta : I \to K$ is any map with dense range, then there is a unique continuous surjection $\beta\eta : \beta I \to K$ such that $\beta\eta(\delta_i) = \eta(i)$, i.e. the following diagram commutes.

$$\begin{tikzcd} \beta I \arrow[r, twoheadrightarrow, "\beta\eta"] & K \\ I \arrow[u, hook, "i \mapsto \delta_i"] \arrow[ur, "\eta"'] & \end{tikzcd}$$