Character space of $L^1(\mathbb{R})$.

Recall that

$$\mathcal{C}_c(\mathbb{R}) = \bigcup_{N>0} \mathcal{C}_c(-N, N) \text{ is dense in } L^1(\mathbb{R}). \qquad (\dagger)$$

Theorem. *$\Gamma_{L^1(\mathbb{R})} = \{\gamma_y : y \in \mathbb{R}\}$, where each $\gamma_y(f) = \int_{\mathbb{R}} f(x)e^{ixy}\, dx$. Moreover, $y \mapsto \gamma_y : \mathbb{R} \to \Gamma_{L^1(\mathbb{R})}$ is a homeomorphism.*

Proof. We recall duality relation $L^1(\mathbb{R})^* \cong L^\infty(\mathbb{R})$. Hence if we fix $\gamma \in \Gamma_{L^1(\mathbb{R})}$, then there is χ in $L^\infty(\mathbb{R})$ for which $\gamma(f) = \int_{\mathbb{R}} f\chi$ for f in $L^1(\mathbb{R})$. Then, using Fubini's Theorem and left invariance of Haar measure, we have that

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} f(x)g(y)\chi(x)\chi(y)\, d(x,y) &= \gamma(f)\gamma(g) = \gamma(f * g) \\ = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y-x)\chi(y)\, dx\, dy &= \int_{\mathbb{R}\times\mathbb{R}} f(x)g(y)\chi(x+y)\, d(x,y)\end{aligned}$$

and hence, $\chi(x)\chi(y) = \chi(x+y)$ for a.e. (x,y) in $\mathbb{R}\times\mathbb{R}$. Now if we let $s * f(y) = f(y-s)$ for s, y in $\mathbb{R}$ and f in $L^1(\mathbb{R})$, then for a.e. s, if f is such that $\gamma(f) = 1$ we have

$$\gamma(s * f) = \int_{\mathbb{R}} f(y-s)\chi(y)\, dy = \int_{\mathbb{R}} f(y)\chi(s+y)\, dy = \chi(s)\gamma(f) = \chi(s).$$

Since $s \mapsto s * f : \mathbb{R} \to L^1(\mathbb{R})$ is continuous [follows form ($\dagger$)], $s \mapsto \gamma(s * f) =: \chi_f(s)$ is continuous, and hence $\chi = \chi'$ a.e.; in particular letting $\chi = \chi_f$, we have $\chi(x+y) = \chi(x)\chi(y)$ for all x, y. Also we note that $\|\chi\|_\infty = \|\gamma\| \leq 1$.

We have that $\mathcal{C}_c^2(\mathbb{R}) = \bigcup_{N>0} \mathcal{C}_c^2(-N, N)$ is dense in $L^1(\mathbb{R})$ [use the fact that each $\mathcal{C}_c^2(-N, N)$ is uniformly dense in $\mathcal{C}_c(-N, N)$ (Stone-Weierstrauss), the fact that $\|\cdot\|_\infty \leq 2N\|\cdot\|_1$ on each $\mathcal{C}_c(-N, N)$, and then ($\dagger$)]. Hence we can find u in $\mathcal{C}_c^2(\mathbb{R})$ for which $\gamma(u) = 1$. Thus for χ as above, we have for each s in $\mathbb{R}$ that

$$\chi(s) = \chi(s)\gamma(u) = \int_{\mathbb{R}} u(x)\chi(s+x)\, dx = \int_{\mathbb{R}} u(x-s)\chi(x)\, dx.$$

Now for y, h in $\mathbb{R}$ and $h \neq 0$, two applications of the mean value theorem provide $h(y)$ between h and 0 for which

$$\left|\frac{u(y-h)-u(y)}{h}+u'(y)\right| = |-u'(y+h(y))+u'(y)| \leq \|u''\|_\infty |h(y)| \leq \|u''\|_\infty |h|.$$

Hence, letting $N > 0$ be so $\mathrm{supp}(u) \subset (-N, N)$ and using the fact that $\|\chi\|_\infty \leq 1$ and the last two equations, we have

$$\begin{aligned}
&\left|\frac{\chi(s+h)-\chi(s)}{h}+\int_{\mathbb{R}} s * u'(x)\chi(x)\,dx\right| \\
&\qquad \leq \int_{\mathbb{R}}\left|\frac{u(x-s-h)-u(x-s)}{h}+u'(x-s)\right| dx \\
&\qquad \leq \int_{-N+s-|h|}^{N+s+|h|} \|u''\|_\infty |h|\,dx = 2(N+|h|)\|u''\|_\infty |h| \xrightarrow{h\to 0} 0
\end{aligned}$$

so χ is differentiable. Furthermore, we have

$$\chi'(s) = \lim_{h\to 0}\frac{\chi(s+h)-\chi(s)}{h} = \lim_{h\to 0}\frac{\chi(h)-\chi(0)}{h}\chi(s) = \chi'(0)\chi(s).$$

Let $y = i\chi'(0)$ and we have

$$\frac{d}{ds}[\chi(s)e^{-iys}] = (\chi'(0)-iy)\chi(s)e^{-iys} = 0$$

while $\chi(0)e^{iy\cdot 0} = 1$, so $\chi(s) = e^{its}$ for all s. Thus we get that $\gamma = \gamma_y$, as claimed.

If $y \neq y'$, then $\gamma_y \neq \gamma_{y'}$. Indeed, $\chi_y(x) = e^{iyx}$ and $\chi_{y'}(x) = e^{iy'x}$ define distinct continuous bounded functions, hence distinct elements of $L^\infty(\mathbb{R})$. Furthermore each γ_y is easily checked to define an element of $\Gamma_{L^1(\mathbb{R})}$.

Now let us check the continuity of $y \mapsto \gamma_y$. We note first that $y \mapsto e^{iyx}$ is uniformly continuous for x on compact subsets of $\mathbb{R}$. Thus for f in $L^1(\mathbb{R})$ and $\epsilon > 0$, we can use dominated convergence to find N for which $g = f1_{(-N,N)}$ satisfies $\|f-g\|_1 < \epsilon$. Now

$$\begin{aligned}
|\gamma_y(f)-\gamma_{y'}(f)| &\leq |\gamma_y(f)-\gamma_y(g)| + |\gamma_y(g)-\gamma_{y'}(g)| + |\gamma_{y'}(g)-\gamma_{y'}(f)| \\
&\leq 2\|f-g\|_1 + \int_{\mathbb{R}} |g(x)||e^{ixy}-e^{ixy'}|\,dx \\
&< 2\epsilon + \|g\|_1 \sup_{-N\leq x\leq N}|e^{ixy}-e^{ixy'}| \xrightarrow{y'\to y} 0.
\end{aligned}$$

Hence it follows by arbitrariness of ϵ that $|\gamma_y(f) - \gamma_{y'}(f)| \xrightarrow{y' \to y} 0$.

Now we consider the **Riemann-Lebesgue Lemma**: *for f in $L^1(\mathbb{R})$ we have*

$$\lim_{|y| \to \infty} \gamma_y(f) = \int_{\mathbb{R}} f(x) e^{iyx} \, dx = 0.$$

[Prove this by estimating f by simple functions.] From this it follows that $y \mapsto \gamma_y : \mathbb{R}_\infty \to \Gamma_{L^1(\mathbb{R})} \cup \{0\}$ (as usual, $\gamma_\infty = 0$) is continuous. This map being bijective, and continuous, is a homeomorphism. Hence $y \mapsto \gamma_y : \mathbb{R} \to \Gamma_{L^1(\mathbb{R})}$ is a homeomorphism. $\square$

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