C*-ALGEBRAS ARE SEMI-SIMPLE

In class we saw the following.

GNS Theorem. *Given a C*-algebra $\mathcal{A}$, there is an injective and completely reducible $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.*

It is straighforward to reinterpret this as saying that there is a family $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i \in I$ of irreducible $*$-representations such that $\{0\} = \bigcap_{i\in I} \ker \pi_i$. Hence, morally speaking a C*-algebra is semisimple. Notice that weird things happen, though. The self representation of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ is clearly irreducible, yet $\mathcal{B}(\mathcal{H})$ admits an non-trivial ideal, $\mathcal{K}(\mathcal{H})$, so $\mathcal{B}(\mathcal{H})$ is not *simple*. Hence there is no infinite dimensional analogue of the Artin-Wedderburn Theorem for $\mathbb{C}$-algebras, which entials that the quotient of an algebra by an irreducible representation is simple. It is still true that the for a C*-algebra annihilators of all simple modules (in the sense of algebra), which may be deduced from something called Kadison's Transitivity Theorem (see Wikipedia, for example).

However, let us reduce this to the setting which would arise naturally in an abstract algebra course, and yields a very simple proof.

Definition. Let $\mathcal{A}$ be a $\mathbb{C}$-algebra. A left ideal $\mathcal{L}$ is called *maximal* if $\{0\} \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and there is no left ideal $\mathcal{L}'$ for which $\mathcal{L} \subsetneq \mathcal{L}' \subsetneq \mathcal{A}$. We let $\Lambda(\mathcal{A}) = \{\mathcal{L} \subset \mathcal{A} : \mathcal{L} \text{ is a maximal left ideal}\}$. We the let the *Jacobsen radical* of $\mathcal{A}$ be given by

$$\mathrm{rad}\mathcal{A} = \bigcap_{\mathcal{L}\in\Lambda(\mathcal{A})} \mathcal{L}.$$

If $\Lambda(\mathcal{A}) = \varnothing$, we let $\mathrm{rad}\mathcal{A} = \{0\}$. If $\mathcal{A}$ is non-unital, we set $\mathrm{rad}\mathcal{A} = \mathcal{A} \cap \mathrm{rad}\widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is the unitization. If $\mathrm{rad}\mathcal{A} = \{0\}$, we say that $\mathcal{A}$ is semi-simple.

Hence, if $\mathcal{A}$ is commutative, then $\mathrm{rad}\mathcal{A}$ is simply the intersection of all maximal modular ideals. There is a right version of this too, which is equivalent to the left version. For simplicity, let us stick to my left-wing prejudices.

Examples. (i) $\Lambda(\mathbb{C}) = \varnothing$, so $\mathrm{rad}\mathbb{C} = \{0\}$.

(iii) We consider $\widetilde{\mathcal{K}} = \mathcal{K}(\mathcal{H}) + \mathbb{C}I$, for any Hilbert space. (Notice that if $d = \dim \mathcal{H} < \infty$, then this is simply $\mathrm{M}_d(\mathbb{C})$, and the addition of I is superfluous.) One can show that $\Lambda(\widetilde{\mathcal{K}}) \supseteq \{\widetilde{\mathcal{K}}(I - \langle\cdot, x\rangle x) : x \in \mathcal{H}, \|x\| =$

1}, i.e. spaces of operators whose range misses exactly a one-dimensional subspace. If $\dim \mathcal{H} < \infty$ then this describes all of $\Lambda(\widetilde{\mathcal{K}})$, otherwise we must add the maximal ideal $\mathcal{K}(\mathcal{H})$, too. But then we see that

$$\mathrm{rad}\mathcal{K}(\mathcal{H}) \subseteq \bigcap \{\mathcal{K}(\mathcal{H})(I - \langle \cdot, x\rangle x) : x \in \mathcal{H}, \|x\| = 1\} = \{0\}$$

so this algebra is semi-simple. Of course, this result will also follow from a theorem, below.

(iii) Let $\mathcal{T}(n) = \{T \in \mathrm{M}_n(\mathbb{C}) : T_{ij} = 0 \text{ for } i > j\}$. Then $\Lambda(\mathcal{T}(n))$ consists of the ideals $\mathcal{L}_i = \{T \in \mathcal{T}(n) : T_{ii} = 0\}$, so $\mathrm{rad}\mathcal{T}(n) = \mathcal{T}_0(n) = \{T \in \mathcal{T}(n) : T_{ii} = 0, i = 1, \dots, n\}$.

(iv) (a commutative example, for old times' sake) Recall that in $\mathcal{C}^1[0,1]$ we have failure of spectral synthesis at $\{\frac{1}{2}\}$: $\mathcal{I} = \{f : f(\frac{1}{2}) = 0 = f'(\frac{1}{2})\} \subsetneq \mathrm{k}(\{\frac{1}{2}\})\}$. Consider the algebra $\mathcal{A} = \mathrm{k}(\{\frac{1}{2}\})/\mathcal{I}$. Notice that if $f \in \mathrm{k}(\{\frac{1}{2}\})$ then $(f^2)' = 2ff'$ so $f^2 \in \mathcal{I}$. Thus, for each a in $\mathcal{A}$ we have $a^2 = 0$. Thus we admit no characters, hence no modular ideals. Thus $\mathrm{rad}\mathcal{A} = \mathcal{A}$.

Lemma. *If $\mathcal{A}$ is unital and $a \in \mathrm{rad}\mathcal{A}$, then $e - a$ admits a left inverse.*

Proof. No left ideal admits e, and hence a standard Zorn's lemma argument tells us that any left ideal is contained in a maximal left ideal. Thus if $e - a$ admits no left inverse, so $\mathcal{A}(e - a) \subset \mathcal{L}$ for some $\mathcal{L}$ in $\Lambda(\mathcal{A})$. If we assume that $a \in \mathrm{rad}\mathcal{A}$ then $e = a + (e - a) \in \mathrm{rad}\mathcal{A} + \mathcal{L} = \mathcal{L}$, contradicting that $\mathcal{L}$ is an ideal. □

Corollary. *If $\mathcal{A}$ is not unital, then any element of $\mathrm{rad}\mathcal{A}$ is left advertible, i.e. there is b in $\mathcal{A}$ for which $ba - b - a = 0$.*

Proof. In $\widetilde{\mathcal{A}}$, any left inverse of $e - a$ must be of the form $e - b$, where b is a left adverse of a. □

Theorem. *If $\mathcal{A}$ is a C^*-algebra, then $\mathrm{rad}\mathcal{A} = \{0\}$, i.e. $\mathcal{A}$ is semi-simple.*

Proof. We may and will assume that $\mathcal{A}$ is unital. We first observe that if $a \in \mathcal{A}_h$ admits a left inverse b, then $ab^* = (ba)^* = e^* = e$ so $b = be = bab^* = eb^* = b^*$, and hence $b = b^{-1}$, i.e. $a \in \mathrm{GL}(\mathcal{A})$. Now if a is a general element of $\mathcal{A} \setminus \{0\}$ we have that $r(a^*a) = \|a^*a\| = \|a\|^2 > 0$ and there is $\lambda \in \sigma(a^*A) \setminus \{0\}$ so $e - \frac{1}{\lambda}a^*a \notin \mathrm{GL}(\mathcal{A})$. But then it follows the lemma that $\frac{1}{\lambda}a^*a \notin \mathrm{rad}\mathcal{A}$, and, since $\mathrm{rad}\mathcal{A}$ is a left ideal, neither do we have $a \in \mathrm{rad}\mathcal{A}$. □

We may extend semisimplicity to a certain class of algebras.

Corollary. *Suppose $\mathcal{A}$ is an involutive algebra for which there exists an injective $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. then* $\mathrm{rad}\mathcal{A} = \{0\}$.

Proof. We may suppose that $\mathcal{A}$ is unital, using the techniques we saw in class for extending a $*$-representation (see proof of extension of states). We have that $\pi(\mathrm{GL}(\mathcal{A})) \subseteq \mathrm{GL}(\mathcal{H})$. Then for a in $\mathcal{A} \setminus \{0\}$ we see that $\pi(a^*a) = \pi(a)^*\pi(a)$ admits $\lambda \neq 0$ for which $I - \frac{1}{\lambda}\pi(a^*a) \notin \mathrm{GL}(\mathcal{H})$, whence $e - \frac{1}{\lambda}a^*a \notin \mathrm{GL}(\mathcal{A})$. We deduce, as above, that $a \notin \mathrm{rad}\mathcal{A}$. □

Notice that at no point in the proof above, did we require knowing that $\mathcal{A}$ admits a norm.

Example. (i) (Group algebras) Define $\pi : \ell^1(G) \to \mathcal{B}(\mathcal{H})$ by $\pi(f)h = f * h$. Notice that

$$\|f * h\|_2 = \left\|\sum_{t\in G} f(t)h(t^{-1}\cdot)\right\|_2 \leq \sum_{t\in G} |f(t)|\|h(t^{-1}\cdot)\|_2 = \|f\|_1\|h\|_2$$

so π is defined. It is easy the chack that π is linear. Furthermore, using that absolutely convergeing sums are rearrangable we have

$$\begin{aligned}\langle f^* * h, h'\rangle &= \sum_{s\in G}\sum_{t\in G} f(t)h(t^{-1}s)\overline{h'(s)} = \sum_{t\in G}\sum_{s\in G} f(t)h(s)\overline{h'(ts)} \\ &= \sum_{s\in G}\sum_{t\in G} h(s)\overline{\overline{f(t^{-1})}h'(t^{-1}s)} = \langle h, f^* * h'\rangle\end{aligned}$$

so $\pi(f^*) = \pi(f)^*$. Notice that

$$\ker\pi \subseteq \left\{f \in \ell^1(G) : f = f * \delta_e = \pi(f)\delta_e = 0\right\} = \{0\}$$

so π is injective. Hence $\ell^1(G)$ is semisimple. Notice that even for G abelian, this fact is not obvious, i.e. that $\Gamma_{\ell^1(G)}$ separates points. We also obtain that for any group G, the complex group ring $\mathbb{C}[G]$ is a semisimple algebra.

(ii) (A continuous group algebra) Define $\pi : L^1(\mathbb{R}) \to \mathcal{B}(L^2(\mathbb{R}))$ by $\pi(f)h = f * h$. For f in the dense subspace $\mathcal{C}_c(\mathbb{R})$ of $L^1(\mathbb{R})$, say $\mathrm{supp} f \subseteq [-N, N]$ we may regard $\pi(f)h = \int_{-N}^{N} f(t)h(t^{-1}\cdot)$ as an $L^2(\mathbb{R})$-valued Riemann integral. By integral versions of the calculations in (i) we see that $\|\pi(f)\| \leq \|f\|_1$ and

$\pi(f^*) = \pi(f)^*$, where $f^*(t) = \overline{f(-t)}$ for a.e. t in $\mathbb{R}$. A standard approximation argument shows that these hold for any f in $L^1(\mathbb{R})$. The sequence $h_n = \frac{1}{2n}1_{[-n,n]}$ in $L^1 \cap L^2(\mathbb{R})$ satisfies that

$$\lim_{n\to\infty} \|f * h_n - f\|_1 = 0$$

as may be verified by a standard argument in measure theory, and hence if $f \neq 0$, $f * h_n \neq 0$ for sufficiently large n. It follows that

$$\ker \pi \subseteq \bigcap_{n=1}^{\infty} \{f \in L^1(\mathbb{R}) : f * h_n = 0\} = 0.$$

Hence $L^1(\mathbb{R})$ is semisimple. Even the dense convolution subalgebra $(\mathcal{C}_c(\mathbb{R}), *)$ is semisimple.