Mutivairable Analytic Functions

Let us fix a positive integer n and an open set $U \subset \mathbb{R}^n$. Given a fixed c in U, we define a *(multivariable) power series* about c to be any series of the form

$$f(x) = \sum_{\kappa \in \mathbb{N}_0^n} \alpha_\kappa (x-c)^\kappa \tag{PS}$$

for some choice of $(\alpha_\kappa)_{\kappa \in \mathbb{N}_0^n} \subset \mathbb{R}$. Some order of explaination of natation is required. $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the set of non-negative integers and elements $\kappa = (\kappa_1, \dots, \kappa_n)$ of $\mathbb{N}_0^n$ are called multiindices. For such a multiindex κ we let and $x \in \mathbb{R}^n$, we let

$$x^\kappa = x_1^{\kappa_1} \dots x_n^{\kappa_n}.$$

The notion of convergence of a power series is not as straighforward as in the one-variable case. In the defintion below, we remark that all series of non-negative entries are convergent independant of arbitrary permutations of the indices of convergence.

Definition. A function $f : U \to \mathbb{R}$ is *analytic* if for each c in U there is power series as in (PS) for all x in a neighbourhood of c, and this power series is absolutely convergent

$$\sum_{\kappa \in \mathbb{N}_0^n} |\alpha_\kappa||(x-c)^\kappa| < \infty.$$

Now let us refine our knowledge of these. For r in $\mathbb{R}^n$ we write

$$r > 0 \text{ if } r_1 > 0, \dots, r_n > 0$$

and let $\mathbb{R}_{>0}^n$ denote the set of such pointwise positive elements.

Abel's Lemma. *Given U, f, c and the power series (PS) above, there is r in $\mathbb{R}_{>0}^n$ for which*

$$\sum_{\kappa \in \mathbb{N}_0^n} |a_\kappa| r^\kappa < \infty. \tag{$\heartsuit$}$$

Hence for any $r' < r$, on $D(c, r') = \{x \in \mathbb{R}^n : |x_i - c_i| \leq r_i' \text{ for } i = 1, \dots, n\}$, the power series (PS) is absolutely uniformly convergent, i.e. any $\epsilon > 0$, there *is a finite $F_\epsilon \subset \mathbb{N}_0^n$ such that for x in $D(c,r) \cap U$*

$$\left| \sum_{\kappa \in \mathbb{N}_0^n \setminus F_\epsilon} a_\kappa (x-c)^\kappa - f(x) \right| < \epsilon.$$

Proof. To begin, we simply pick an x' in the neighbourhood of c specified in the definition above, such that $r_i = |x'_i - c_i| > 0$ for each i. This gives ($\heartsuit$). Observe that the convergence of the series implies that the idividual terms were bounded: $|a_\kappa| r^\kappa \leq M$ for some $M > 0$.

We then observe for any $x \in D(c,r) \cap U$ that for any K in $\mathbb{N}$ that

$$\begin{aligned} \left| \sum_{\substack{\kappa \in \mathbb{N}_0^n \\ \max_{i=1,\ldots,n} \kappa_i < k}} a_\kappa (x-c)^\kappa - f(x) \right| &= \sum_{\substack{\kappa \in \mathbb{N}_0^n \\ \min_{i=1,\ldots,n} \kappa_i \geq k}} |a_\kappa| |(x-c)^\kappa| \\ &= \sum_{\substack{\kappa \in \mathbb{N}_0^n \\ \min_{i=1,\ldots,n} \kappa_i \geq k}} |a_\kappa| r^\kappa \frac{|(x-c)^\kappa|}{r^\kappa} = \sum_{\substack{\kappa \in \mathbb{N}_0^n \\ \min_{i=1,\ldots,n} \kappa_i \geq k}} M \prod_{i=1}^n (r'_i/r_i)^{\kappa_i} \\ &= M \prod_{i=1}^n \frac{(r'_i/r_i)^k}{1 - r'_i/r_i} \quad \overset{k\to\infty}{\longrightarrow} \quad 0. \end{aligned}$$

These estimates are independant of x in $D(c,r') \cap U$, giving uniform convergence. □

We define for a multiindices κ, λ in $\mathbb{N}_0^n$ with $\lambda \leq \kappa$, i.e. $\lambda_i \leq \kappa_i$ for each i

$$\kappa! = \kappa_1! \ldots \kappa_n! \text{ and } \binom{\kappa}{\lambda} = \frac{\kappa!}{\kappa!(\kappa - \lambda)!}.$$

This notation facilitates a generalised binomial theorem

$$(x+y)^\kappa = \sum_{\lambda \leq \kappa} \binom{\kappa}{\lambda} x^{\kappa - \lambda} y^\lambda$$

as may be proved with a simple induction argument.

The following may regarded as a converse to Abel's Lemma, i.e. a reasonbale power series defines and analytic function on a neigbourhood of a point.

Proposition. *Let $(a_\kappa)_{\kappa\in\mathbb{N}_0^n} \subset \mathbb{R}^n$ and r in $\mathbb{R}^n_{>0}$ be such that ($\heartsuit$) holds. Then for c in $\mathbb{R}^n$ for which $|c_i| < r_i$ for each i, we have that the power series (PS) defines an analytic function in a neigbourhood of c.*

Proof. Fix h in $\mathbb{R}^n$ for which $|c_i + h_i| < r_i$ for each i. Then let $r' > 0$ be so that for x with $|x_i - (c_i + h_i)| \leq r'_i$ for each i, $|x_i| < r_i$ too, i.e. choose $r'_i < r_i - |c_i + h_i|$. Then for x in $D(c+h, r')$ we have a power series in x given by

$$f(x) = \sum_{\kappa\in\mathbb{N}_0^n} a_\kappa (x - (c+h))^\kappa = \sum_{\kappa\in\mathbb{N}_0^n} \sum_{\lambda\leq\kappa} \binom{\kappa}{\lambda} a_\kappa x^\lambda [-(c+h)]^{\kappa-\lambda}.$$

This converges absolutely, since

$$\sum_{\kappa\in\mathbb{N}_0^n} \sum_{\lambda\leq\kappa} \binom{\kappa}{\lambda} |a_\kappa||x^\lambda||(c+h)^{\kappa-\lambda}| = \sum_{\kappa\in\mathbb{N}_0^n} |a_\kappa| \begin{bmatrix} |x_1| + |c_1 + h_1| \\ \vdots \\ |x_n| + |c_n + h_n| \end{bmatrix}^\kappa$$

where each $|x_i| + |c_i + h_i| \leq r'_i + |c_i + h_i| < r_i$. $\square$

Remark. Let F be a one-variable analytic function in a neigbourhood of 0, so we have $F(t) = \sum_{k=1}^\infty \alpha_k t^k$. The root test provides radius of convergence $R = [\limsup_{k\to\infty} |\alpha_k|]^{-1}$ (which may be ∞ if the limit is 0).

For each pair i, j form $1, \dots, n$ we obtain a function $f_{ij} : U_R = \{X \in \mathrm{M}_n(\mathbb{R}) : \|X\| < R\} \to \mathbb{R}$ given by $f_{ij}(X) = F(X)_{ij}$. We observe the formula

$$(X^k)_{ij} = \sum_{j_1,\dots,j_k=1}^{n} X_{ij_1} X_{j_1 j_2} \dots X_{j_k j}.$$

We appeal to submultiplicatively of the norm to see that for each $X_{i'j'}$ that

$$|X_{i'j'}| = \|X_{i'j'} E_{i'j'}\| = \|E_{i'i'} X E_{j'j'}\| \leq \|X\|$$

where $E_{i''j''}$ is the matrix with 1 in the i'', j''th position, and zeros elsewhere. Let us write

$$f_{ij}(X) = \sum_{\kappa\in\mathbb{N}_0^{n\times n}} \alpha_{ij,\kappa} X^\kappa$$

where $\mathbb{N}_0^{n\times n}$ denotes the mutiindices indexed by pairs i,j and $X^\kappa = \prod_{i,j=1}^n X_{ij}{}^{\kappa_{ij}}$. Now we develop a develop a crude estimate for X in $U_{R/n}$:

$$\sum_{\kappa\in\mathbb{N}_0^{n\times n}} |\alpha_{ij,\kappa}||X^\kappa| \le \sum_{k=0}^{\infty} |\alpha_{ij,\kappa}| \sum_{j_1,\dots,j_k=1}^{n} |X_{ij_1}X_{j_1j_2}\dots X_{j_kj}| \le |\alpha_{ij,\kappa}|n^k \|X\|^k$$

where the latter series converges by our choice of X. Hence f_{ij} defines an analytic function on a neighbourhood of 0. □

For a mutiindex κ in $\mathbb{N}_0^n$ we let

$$|\kappa| = \kappa_1 + \dots + \kappa_n.$$

Letting $|h|$ denote the norm of h in $\mathbb{R}^n$ we then observe the inequality :

$$|h^\kappa| = |h_1|^{\kappa_1}\dots|h_n|^{\kappa_n} \le |h|^{\kappa_1}\dots|h|^{\kappa_n} = |h|^{|\kappa|}.$$

Theorem. *Any analytic function $f : U \to \mathbb{R}$ (U is an open set in $\mathbb{R}^n$) is differentiable. In fact any directional derivative $D_u f$ of f is itself analytic and hence all higer order directional derivatives are analytic, so f is also $\mathcal{C}^\infty$.*

Proof. We begin with a technical estimate. We let ϵ_j, $j = 1,\dots,n$ denote the basic multiindices with $\epsilon_{jj} = 1$ and 0 otherwise. If in $\mathbb{N}_0^n$, $\kappa \not\geq \lambda$, write $\kappa - \lambda = 0$. Let $x,h \in \mathbb{R}^n$ with $|h| \le 1$ and $\kappa \in \mathbb{N}_0^n$. We compute

$$\left|(x+h)^\kappa - x^\kappa - \sum_{j=1}^{n} \kappa_j x^{\kappa-\epsilon_j} h_j\right| \le \sum_{\substack{\lambda\le\kappa \\ \lambda\notin\{0,\epsilon_1,\dots,\epsilon_n\}}} \binom{\kappa}{\lambda} |x^{\kappa-\lambda}||h|^{|\lambda|}$$

$$\le \sum_{\lambda\le\kappa} |x^{\kappa-\lambda}||h|^{|\lambda|} = \begin{bmatrix} |x_1| + |h| \\ \vdots \\ |x_n| + |h| \end{bmatrix}^\kappa .$$

Now suppose we have (PS) defining f in a neighbourhood of c. Let $r > 0$ be as in Abel's Lemma x be so $|x_i - c_i| < r_i$ and h be so $|x_i - c_i| + |h| < r_i$ for each i. Then our estimate above, with $x - c$ in place of x, and Abel's

Lemma show that

$$\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa|\left|(x+h-c)^\kappa-(x-c)^\kappa-\sum_{j=1}^n\kappa_j(x-c)^{\kappa-\epsilon_j}h_j\right|$$
$$=\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa||h|^2\sum_{\substack{\lambda\leq\kappa\\ \lambda\notin\{0,\epsilon_1,\ldots,\epsilon_n\}}}\binom{\kappa}{\lambda}|(x-c)^{\kappa-\lambda}||h|^{|\lambda|-2}$$
$$\leq\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa|\begin{bmatrix}|x_1-c_1|+|h|\\ \vdots\\ |x_n-c_n|+|h|\end{bmatrix}^\kappa<\infty$$

Moroever, by uniform convergence, this series tends to 0 as $h\to 0$. Hence another application of Abel's Lemma shows that the linear functional

$$h\mapsto Df(x)h=\sum_{j=1}^n\kappa_j\sum_{\kappa\in\mathbb{N}_0^n}x^{\kappa-\epsilon_j}h_j$$

is the derivative of f at x. Indeed, observe identifying this functional with a vector in $\mathbb{R}^n$ we have in the jth coordinate

$$\kappa_j\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa||x^{\kappa-\epsilon_j}|\kappa_j\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa|r^{\kappa-\epsilon_j}=\frac{\kappa_j}{r_j}\sum_{\kappa\in\mathbb{N}_0^n}|\alpha_\kappa|r^\kappa<\infty$$

which by the last propositon this coordinate functional itself is analytic. Moreover, the formula for directional derivatives

$$D_uf(x)=Df(x)u=\sum_{j=1}^n\kappa_j\sum_{\kappa\in\mathbb{N}_0^n}\alpha_\kappa x^{\kappa-\epsilon_j}u_j$$

along with the last proposition, shows that D_uf itself is an analytic function in f. If we apply the same reasoning to D_uf we see that all second order directional derivatives D_vD_uf are analytic, and, inductivly, this applies to all higher order directional derivatives. □

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