PMATH 763, Winter 2017

Assignment #5 Due: April 3

1. The goal of this question is to establish that if G is a compact matrix Lie group, then the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is reductive.

   (a) Show that there is a real inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ for which

   $$([X,Y],Z) = -(Y,[X,Z]) \text{ for } X,Y,Z \in \mathfrak{g};$$

   in other words, for which each $\mathrm{ad}X$ is skew-symmtric. Deduce that $\mathfrak{m} = \mathfrak{z}^\perp$ is a Lie ideal in $\mathfrak{g}$, where $\mathfrak{z} = \mathrm{Z}(\mathfrak{g})$.

   [This is really a version of Maschke's theorem.]

   (b) Show that for $X \in \mathfrak{g}$ that $\mathrm{ad}X$ has purely imaginary eigenvalues, and deduce that the Killing form B on $\mathfrak{g}$ is negative semi-definite, i.e. $B(X,X) \leq 0$ for $X \in \mathfrak{g}$.

   [Consider the realisation $\mathrm{ad}\mathfrak{g} \subset \mathcal{L}(\mathfrak{g}) \cong \mathrm{M}_d(\mathbb{R}) \subset \mathrm{M}_d(\mathbb{C})$ ($d = \dim_\mathbb{R} \mathfrak{g}$), in terms of a basis, orthonormal for $(\cdot,\cdot)$, above.]

   (c) Show that $\mathfrak{z} = \mathfrak{g}^B$ and deduce that $\mathfrak{m}$ is semi-simple.

   Remark. Q3, on A3, tells us that $\mathfrak{m} = [\mathfrak{g},\mathfrak{g}]$, hence we obtain $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$, and $[\mathfrak{g},\mathfrak{g}]$ is semi-simple.

2. Let G be a compact matrix Lie group, and $\pi : G \to \mathrm{U}(\mathcal{V}_\pi)$ and $\sigma : G \to \mathrm{U}(\mathcal{V}_\sigma)$ be two finite dimensional unitary representations of G (i.e. "unitary" with respect to some given inner products). If σ is irreducible, we let $m_{\sigma,\pi}$ denote the *multiplicity* of σ in π, i.e. the number of distinct irreducble subrepresentations of π which are unitarily equivalent to σ. Also let $d_\sigma = \dim_\mathbb{C} \mathcal{V}_\sigma$.

   (a) Show that

   $$\chi_\pi \chi_\sigma = \sum_{\substack{\tau \in \widehat{G} \\ \tau \subset \pi \otimes \sigma}} m_{\tau,\pi\otimes\sigma} \chi_\tau$$

   where $\tau \subset \pi \otimes \sigma$ means that τ is unitarily equivalent to a subrepresentation of $\pi \otimes \sigma$.

   (b) Deduce that if σ is irreducible, then there is a 1-dimensional subspace of $\mathcal{V}_\sigma \otimes \mathcal{V}_{\bar\sigma}$ of fixed points of $\sigma \otimes \bar\sigma$, i.e. points v for which $\sigma \otimes \bar\sigma(g)v = v$ for all g in G.

3. **(a)** Show that for every irreducible representation $\pi : \mathrm{U}(n) \to \mathrm{U}(\mathcal{V})$, $\pi|_{\mathrm{SU}(n)}$ is an irreducible representation of $\mathrm{SU}(n)$.

(b) Show that each irreducible representation $\sigma : \mathrm{SU}(n) \to \mathrm{U}(\mathcal{V})$ extends to an irreducible representation $\tilde{\sigma} : \mathrm{U}(n) \to \mathrm{U}(\mathcal{V})$.

[Determine the form of $\sigma|_{C_n}$, where $C_n = \mathrm{ZSU}(n)$. Write $g = z\bar{z}g$ where $z^n = \det g$.]

(c) Let $\mu, \nu \in \mathbb{Z}^n_+$, and let χ_μ, χ_ν denote the associated characters on $\mathrm{U}(n)$. Show that

$$\chi_\mu|_{\mathrm{SU}(n)} = \chi_\nu|_{\mathrm{SU}(n)} \quad \Leftrightarrow \quad \mu - \nu \in \mathbb{Z}(1, 1, \ldots, 1).$$

Hence deduce that $\widehat{\mathrm{SU}}(n)$ can be parameterized by

$$P = \left\{\mu = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{Z}^{n-1} : \mu_1 \geq \cdots \geq \mu_{n-1} \geq 0\right\}.$$

(d) Deduce that in $\mathrm{SU}(2)$ each irreducible character is determined by

$$\chi_m\left(\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}\right) = z^m + z^{m-2} + \cdots + z^{-m+2} + z^{-m}.$$

(e) Let π_m be the representation associated to χ_m, above. Show that

$$\pi_m \otimes \pi_{m'} \approx \pi_{|m-m'|} \oplus \pi_{|m-m'|+2} \oplus \cdots \oplus \pi_{m+m'-2} \oplus \pi_{m+m'}.$$