PMATH 763, Winter 2017

Assignment #4 Due: March 21

1. Let (X, d) be a metric space. Show that for any compact set $K \subset X$ and any covering of K by open sets $\{V_i\}_{i\in I}$, ther is a *partition of unity* for K with respect to $\{V_i\}_{i\in I}$, i.e. a family $\{f_1, \dots, f_m\}$ of continuous non-negative functions such that

$$\sum_{k=1}^{m} f_k(x) = 1 \text{ for } x \in K, \text{ and each } \mathrm{supp} f_k \in V_{i_k} \text{ for some } i_k \in I.$$

[If $K \subset U$ and $\overline{U} \subset V$, where K is compact and U, V are open, then $f(x) = \min\{\frac{\mathrm{dist}(x,X\setminus U)}{\mathrm{dist}(K,X\setminus U)}, 1\}$ is continuous and satisfies $f|_K = 1$ and $\mathrm{supp} f \subset V$.]

2. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \mathrm{T}_2(\mathbb{R})_0 \cap \mathrm{SL}_2(\mathbb{R})$.

(a) (Iwasawa decomposition) Consider the basis elements

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

for $\mathfrak{sl}_2(\mathbb{R})$. Show that any $g \in G$ admits unique decomposition

$$g = g(\theta, t, s) = u(\theta)a(t)n(s) = \exp(\theta U)\exp(tA)\exp(sN)$$

for $\theta \in [0, 2\pi)$, $s, t \in \mathbb{R}$.

[Begin by establishing the unique decoposition $g = u(\theta)h$ where $h \in H = a(\mathbb{R})n(\mathbb{R})$.]

(b) Show that

$$\{(\varphi_0, G \setminus u(\pi)H), (\varphi_1, G \setminus H)\}$$

where $\varphi_0^{-1} : (-\pi, \pi) \times \mathbb{R}^2 \to G$ and $\varphi_1^{-1} : (0, 2\pi) \times \mathbb{R}^2 \to G$ are given by $\varphi_k^{-1}(\theta, t, s) = g(\theta, t, s)$ $(k = 0, 1)$, defines a $\mathcal{C}^1$-coordinate system on G (it is also $\mathcal{C}^\infty$, but we will only require $\mathcal{C}^1$ in our computations).

(c) Show that an invariant integral on G is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} f\big(g(\theta, t, s)\big)e^{2t}\, d\theta\, dt\, ds.$$

[Hint: $\frac{\partial}{\partial\theta}g(\theta, t, s) = Ug(\theta, t, s)$, $\frac{\partial}{\partial t}g(\theta, t, s) = g(\theta, t, -s)A$, and $\frac{\partial}{\partial s}g(\theta, t, s) = g(\theta, t, s)N$.]

(d) Why is this integral also right invariant?

3. Fix θ in $\mathbb{R}$ and let

$$G_\theta = \left\{ \begin{bmatrix} e^{\theta t}\cos t & e^{\theta t}\sin t & x \\ -e^{\theta t}\sin t & e^{\theta t}\cos t & y \\ 0 & 0 & 1 \end{bmatrix} : t, x, y \in \mathbb{R} \right\}$$

Compute a left invariant integral on G_θ and determine when G_θ is unimodular.

4. The *Cayley transform* $\varphi : \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ is given by $\varphi(z) = \frac{1-z}{1+z}$. It can be easily checked to satisfy $\varphi(i\mathbb{R}) = \mathbb{T} \setminus \{-1\}$ and $\varphi \circ \varphi(z) = z$.

(a) Show that for $f \in \mathcal{C}(\mathbb{T})$ that the normalised invariant integral is given by

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\varphi(it)) \frac{1}{1+t^2} dt.$$

[The integral on the left can be taken for granted; it comes from the coordinate patch $(\log, \mathbb{T} \setminus \{-1\})$ where log is the principal branch of logarithm. Notice that $\{-1\}$ is of Jordan content zero.]

(b) Let $\mathcal{D}_n = \{X \in \mathrm{M}_n(\mathbb{R}) : \det(I + X) \neq 0\}$ and define $\varphi : \mathcal{D}_n \to \mathcal{D}_n$ by

$$\varphi(X) = (I - X)(I + X)^{-1}.$$

Show that φ is a diffeomorphism with derivative $D\varphi(X) \in \mathcal{L}(\mathrm{M}_n(\mathbb{R}))$ given by

$$D\varphi(X)Y = -2(I + X)^{-1} Y (I + X)^{-1},\ X \in \mathcal{D}_n(\varphi),\ Y \in \mathrm{M}_n(\mathbb{R}).$$

(c) Show that $\mathfrak{so}(n) \subset \mathcal{D}_n$ and that $\varphi(\mathfrak{so}(n))$ is an open subset of $\mathrm{SO}(n)$ so that $\big(\varphi, \mathcal{D}_n(\varphi) \cap \mathrm{SO}(n)\big)$ is a $\mathcal{C}^1$-coordinate patch on $\mathrm{SO}(n)$.

(d) If $A \in \mathrm{GL}_n(\mathbb{R})$, show that the map $T_A : \mathfrak{so}(n) \to \mathfrak{so}(n)$ given by $T_A(X) = AXA^{\mathrm{T}}$ has $|\det T_A| = |\det A|^{n-1}$.

[Use polar decomposition.]

(e) Show that for $f \in \mathcal{C}_c(\mathcal{D}_n \cap \mathrm{SO}(n))$

$$\int_{\mathfrak{so}(n)} f(\varphi(X)) \frac{1}{|\det(I + X)|^{n-1}} \prod_{1 \le i < j \le n} dx_{ij}$$

where $X = \sum_{1 \le i < j \le n} x_{ij}(E_{ij} - E_{ji})$, defines an invariant integral on $\mathrm{SO}(n)$.

Note: $\mathrm{SO}(n) \setminus \mathcal{D}_n$ is of Jordan content zero in $\mathrm{SO}(n)$, so the restriction of f being supported on $\mathcal{D}_n \cap \mathrm{SO}(n)$ may be relaxed, in practice.